Weak Turbulence for a Vibrating Plate: Can One Hear a Kolmogorov Spectrum?

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We study the long-time evolution of waves of a thin elastic plate in the limit of small deformation so that modes of oscillations interact weakly. According to the theory of weak turbulence (successfully applied in the past to plasma, optics, and hydrodynamic waves), this nonlinear wave system evolves at long times with a slow transfer of energy from one mode to another. We derive a kinetic equation for the spectral transfer in terms of the second order moment. We show that such a theory describes the approach to an equilibrium wave spectrum and represents also an energy cascade, often called the Kolmogorov-Zakharov spectrum. We perform numerical simulations that confirm this scenario.

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Introduction.—For more than 40 years it has been established that long-time statistical properties of a random fluctuating wave system possess a natural asymptotic closure because of the dispersive nature of the waves and of the weakly nonlinear interaction between them [1,2]. This "weak turbulence theory" has been shown to be a powerful method for studying the evolution of nonlinear dispersive wave systems [3,4]. It follows that the long-time dynamics is driven by a kinetic equation for the distribution of spectral densities. This method has been applied to surface gravity waves [1,5], capillary waves [6], plasma waves [7], and nonlinear optics [8] for instance. The actual kinetic equation has nonequilibrium properties similar to the usual Boltzmann equation for dilute gases, conserving energy and momentum, and it exhibits an H theorem driving the system to equilibrium, characterized by the Rayleigh-Jeans distribution. Most important, besides the elementary equilibrium (or thermodynamic) solution, Zakharov has shown [7] that power-law nonequilibrium solutions also arise, namely, the Kolmogorov-Zakharov (KZ) solutions or KZ spectra, which describe the exchange of conserved quantities (e.g., energy) between large and small length scales.

Experimental evidence of KZ spectra have been found in the ocean surface [9] and in capillary surface waves [10–12]. Numerical simulations have also shown the existence of KZ spectra in weak turbulent capillary waves [13] and, more recently, in gravity waves [14].

In this Letter an oscillating thin elastic plate is considered [15]. Adding inertia to the well known (static) theory of thin plates, one finds the existence of ballistic dispersive waves [16]. They interact via the nonlinear terms that are weak if the plate deformations are small. Understanding the interaction between these waves is thus crucial to describe acoustical properties of the plates. In fact, nonlinear solitary waves have been observed on the surface of a cylindrical shell that show balance between nonlinear effects and dispersion [17]. However, we develop here the first weak turbulence theory for the surface deflection on plate dynamics. We find that the bending waves travel randomly through the system and interact resonantly between each other via the weak nonlinearities. The mathematics behind the resonant condition is formally identical to the conservation of energy and momentum in a classical gas. In this sense an elastic plate is formally equivalent to a 2D gas of classical particles interacting with a nontrivial scattering cross section. An isolated system evolves from a random initial condition to a situation of statistical equilibrium as a gas of particles does. In addition to statistical equilibrium for isolated systems, the weak turbulence theory predicts here an energy cascade from a source of energy (a driving forcing) to a dissipation scale typical of irreversible processes.

More precisely, we have in mind an elastic thin plate under an external low frequency (few times the slowest plate mode) random forcing. Typically the gravest mode for a $10 \times 10\ \mathrm{cm}^2$ free bounded steel sheet of 0.1 mm thick is about 50 Hz and is a bit higher for a clamped sheet. Internal resonance among modes buildup an energy cascade from the injection scale to small scales where it is ultimately dissipated mostly because of the boundaries, the air entraintement, viscoelastic flows, or heat transfer. A genuine cascade should setup if dissipation occurs at small scales only. One needs to be careful concerning the heat transfer since the damping coefficient does not depend on the oscillation frequency there. However, heat loss is a weak effect and can be in general neglected (see section 30 in Ref. [18]): indeed, for the above example, the heat loss time scale is about 15 times smaller than the slowest oscillation at room temperature. As in fluids, viscosity in solids acts only at small scale. Finally, in a real experiment the boundaries play an important role because of the finite value of the experimental setup impedance. Such a damping coefficient grows linearly with wave number and is probably the most relevant source of dissipation. Therefore, it seems possible that energy cascades from the scale of the plate to the dissipation scale.

Moreover, while there is often a lack of direct observations of weak turbulence predictions, we exhibit numerically relaxation to equilibrium and energy cascade for the plate dynamics, confirming the scenario presented above. The plate dynamics is illustrated in Fig. 1 for an isolated system where the plate deformations are shown at initial time and after a long evolution.

Theory.—The starting point is the dynamical version of the Föppl–von Kármán equations [18] for the plate deformation $\zeta(x, y, t)$ and the stress function $\chi(x, y, t)$:

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{Eh^2}{12(1-\sigma^2)} \Delta^2 \zeta + \{\zeta, \chi\}; \tag{1}$$

$$\frac{1}{E} \Delta^2 \chi = -\frac{1}{2} \{\zeta, \zeta\}, \tag{2}$$

where h is the thickness of the elastic sheet. The material has a mass density ρ, a Young's modulus E, and a Poisson ratio σ. $\Delta = \partial_{xx} + \partial_{yy}$ is the usual Laplacian and the bracket $\{\cdot, \cdot\}$ is defined by $\{f, g\} \equiv f_{xx} g_{yy} + f_{yy} g_{xx} - 2 f_{xy} g_{xy}$, which is an exact divergence, so Eq. (1) preserves the momentum of the center of mass, namely $\partial_{tt} \int \zeta(x, y, t) dx dy = 0$. The first term on the right-hand side of (1) represents the bending while the second one $\{\zeta, \chi\}$, together with Eq. (2), represents the stretching [19].

Despite the complexity of Eqs. (1) and (2) the system presents a Hamiltonian structure that is straightforward in Fourier space. Defining the Fourier transforms as $\zeta_{\boldsymbol{k}}(t) = \frac{1}{2\pi} \int \zeta(\boldsymbol{x}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d^2\boldsymbol{x}$ (with $\zeta_{\boldsymbol{k}} = \zeta^*_{-\boldsymbol{k}}$), then one gets from Eq. (2): $\chi_{\boldsymbol{k}}(t) = -\frac{E}{2|\boldsymbol{k}|^4} \{\zeta, \zeta\}_{\boldsymbol{k}}$, where $\{\zeta, \zeta\}_{\boldsymbol{k}}$ is the Fourier transform of $\{\zeta, \zeta\}$. The final equation is a nonlinear oscillator with the usual ballistic dispersion relation of bending waves $\omega_{\boldsymbol{k}} = hc|\boldsymbol{k}|^2 = hck^2$ [$c = \sqrt{E/12(1-\sigma^2)\rho}$ has the dimension of a velocity] [16,18]:

$$\rho \frac{\partial^2 \zeta_{\boldsymbol{k}}}{\partial t^2} = -\frac{Eh^2 k^4}{12(1-\sigma^2)} \zeta_{\boldsymbol{k}} - \int V_{-\boldsymbol{k},\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \zeta_{\boldsymbol{k}_4} \delta^{(2)}(\boldsymbol{k} - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) d^2\boldsymbol{k}_{2,3,4},$$

where $V_{12;34} = \frac{E}{2(2\pi)^2} \frac{|\boldsymbol{k}_1 \times \boldsymbol{k}_2|^2 |\boldsymbol{k}_3 \times \boldsymbol{k}_4|^2}{|\boldsymbol{k}_1 + \boldsymbol{k}_2|^4}$ and $d^2\boldsymbol{k}_{2,3,4} \equiv d^2\boldsymbol{k}_2 d^2\boldsymbol{k}_3 d^2\boldsymbol{k}_4$. The Hamiltonian structure becomes evident if we define as canonical variables the deformation $\zeta_{\boldsymbol{k}}(t)$ and the momentum $p_{\boldsymbol{k}}(t) = \rho \partial_t \zeta_{\boldsymbol{k}}(t)$. Finally, the canonical transformation $\zeta_{\boldsymbol{k}} = \frac{X_k}{\sqrt{2}}(A_{\boldsymbol{k}} + A^*_{-\boldsymbol{k}})$ and $p_{\boldsymbol{k}} = -\frac{i}{\sqrt{2} X_k}(A_{\boldsymbol{k}} - A^*_{-\boldsymbol{k}})$ with $X_k = \frac{1}{\sqrt{\omega_k \rho}}$ allows us to write the wave equation in a diagonalized form: $\frac{dA_{\boldsymbol{k}}}{dt} + i\omega_k A_{\boldsymbol{k}} = iN_3(A_{\boldsymbol{k}})$, where $N_3(\cdot)$ is the cubic nonlinear term.

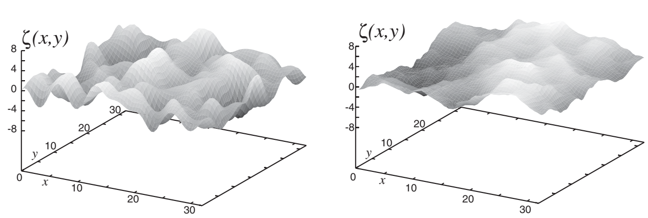

FIG. 1. Zoom over a portion of the surface plate deflection $\zeta(x, y)$. The left-hand image is the initial condition while the right-hand one represents a long-time evolution of the plate.

Weak turbulence theory.—This nonlinear oscillator has two distinct time scales, the rapid oscillation $i\omega_k A_{\boldsymbol{k}}$ and the weak nonlinearity: $iN_3(A_{\boldsymbol{k}})$. Then, following the approach of [4], one changes $A_{\boldsymbol{k}} = a_{\boldsymbol{k}}(t) e^{-i\omega_k t}$ which removes the rapid linear oscillating term:

$$\frac{da^s_{\boldsymbol{k}}}{dt} = -is \sum_{s_1 s_2 s_3} \int J_{-\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} e^{it(s\omega_k - s_1\omega_{k_1} - s_2\omega_{k_2} - s_3\omega_{k_3})} \times a_1^{s_1} a_2^{s_2} a_3^{s_3} \delta^{(2)}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}) d^2\boldsymbol{k}_{123}, \tag{3}$$

where we define $a^s_{\boldsymbol{k}}$ with the two possible choices $s = +$ or $-$ relative to the propagation direction, such that $a^+_{\boldsymbol{k}} \equiv a_{\boldsymbol{k}}$ while $a^-_{\boldsymbol{k}} \equiv a^*_{-\boldsymbol{k}}$. The interaction term reads: $J_{k_1,k_2;k_3,k_4} = \frac{1}{6} X_{k_1} X_{k_2} X_{k_3} X_{k_4} \mathcal{P}_{234} V_{\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4}$, where $\mathcal{P}_{234}$ is the sum over the six possible permutations between 2, 3, and 4. The next step consists of writing a hierarchy of linear equations for the averaged moments: $\langle a^{s_1}_{\boldsymbol{k}_1} a^{s_2}_{\boldsymbol{k}_2} \rangle$, $\langle a^{s_1}_{\boldsymbol{k}_1} a^{s_2}_{\boldsymbol{k}_2} a^{s_3}_{\boldsymbol{k}_3} a^{s_4}_{\boldsymbol{k}_4} \rangle$, etc. A multiscale analysis provides a natural asymptotic closure for higher moments: the fast oscillations drive the system close to Gaussian statistics and higher moments are written in terms of the second order moment: $\langle a_{\boldsymbol{k}_1} a^*_{\boldsymbol{k}_2} \rangle = n_{\boldsymbol{k}_1} \delta^{(2)}(\boldsymbol{k}_1 + \boldsymbol{k}_2)$, where $n_{\boldsymbol{k}}$ is called the wave spectrum.

The wave spectrum thus satisfies a Boltzmann-type kinetic equation describing a slow exchange of energy from one mode to another through four waves resonance:

$$\frac{dn_{\boldsymbol{p}_1}}{dt} = 12\pi \int |J_{\boldsymbol{p}_1\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}|^2 \sum_{s_1 s_2 s_3} n_{\boldsymbol{k}_1} n_{\boldsymbol{k}_2} n_{\boldsymbol{k}_3} n_{\boldsymbol{p}_1} \left(\frac{1}{n_{\boldsymbol{p}_1}} + \frac{s_1}{n_{\boldsymbol{k}_1}} + \frac{s_2}{n_{\boldsymbol{k}_2}} + \frac{s_3}{n_{\boldsymbol{k}_3}} \right) \delta(\omega_{p_1} + s_1\omega_{k_1} + s_2\omega_{k_2} + s_3\omega_{k_3}) \times \delta^{(2)}(\boldsymbol{p}_1 + s_1\boldsymbol{k}_1 + s_2\boldsymbol{k}_2 + s_3\boldsymbol{k}_3) d^2\boldsymbol{k}_{123}. \tag{4}$$

As for the usual Boltzmann equation, Eq. (4) conserves "formally" [20] the total momentum per unit area $\boldsymbol{P} = h \int \boldsymbol{k} n_{\boldsymbol{k}}(t) d^2\boldsymbol{k}$ and the kinetic energy per unit area $\mathcal{E} = h \int \omega_k n_{\boldsymbol{k}}(t) d^2\boldsymbol{k}$, and exhibits an H theorem: let $\mathcal{S}(t) = \int \ln(n_{\boldsymbol{k}}) d^2\boldsymbol{k}$ be the nonequilibrium entropy, then $d\mathcal{S}/dt \geq 0$, for increasing time. However, despite the four waves interaction type kinetic Eq. (4), the "wave action" $\mathcal{N} = \int n_{\boldsymbol{k}}(t) d^2\boldsymbol{k}$ is not conserved. The kinetic Eq. (4) describes thus an irreversible evolution of the wave spectrum towards the Rayleigh-Jeans *equilibrium* distribution which reads, when $\boldsymbol{P} = 0$:

$$n^{\mathrm{eq}}_{\boldsymbol{k}} = T/\omega_k; \tag{5}$$

here T is called, by analogy with thermodynamics, the "temperature" (with units of energy/length, i.e., a force), which is naturally related to the initial energy by $\mathcal{E}_0 = h \int \omega_k n_{\mathrm{eq}} d^2\boldsymbol{k} = hT \int d^2\boldsymbol{k}$. The quantity $\int d^2\boldsymbol{k}$ is the number of degrees of freedom per unit surface. Therefore each degree of freedom takes the same energy: hT. Naturally, for an infinite system this number diverges (as well as the energy). This classical Rayleigh-Jeans catastrophe is always suppressed due to some physical cutoff discussed above. Numerical simulations on regular grid provide also a natural cutoff $k_c = \pi/dx$, where dx is the mesh size, which gives $\mathcal{E}_0 = \pi h T k_c^2$ for a large plate.

Kolmogorov spectrum.—In addition, isotropic nonequilibrium distribution solutions can arise [7]. They have a major importance in the nonequilibrium process for the energy transfer between different scales. These solutions can be guessed via a dimensional analysis argument but they are, in fact, exact solutions of the kinetic equation. Despite some differences with the usual kinetics equation, the Zakharov method can be applied here. Assuming an isotropic spectrum $n_{\boldsymbol{k}} \equiv n_{|\boldsymbol{k}|}$ and integrating over the angles the scattering amplitude $|J_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}|^2\delta^{(2)}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)$, the new scattering amplitude depends only on the modulus $k_i = |\boldsymbol{k}_i|$, and it can be written as a function of the frequencies ω_{k_i}: $S_{\omega_1,\omega_2,\omega_3,\omega_4} = \frac{1}{6}\mathcal{P}_{234}\int \frac{|J_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}|^2}{|\boldsymbol{k}_2\times\boldsymbol{k}_3|}d\varphi_4$. Since the degree of homogeneity of $|J|^2$ in k is zero, S scales as $1/k^2 \sim 1/\omega_k$.

Looking for a power-law solution $n_k = A\omega_k^{-\alpha}$, the eight terms of the collisional integral in the right-hand side of (4) decompose into $3\mathrm{Coll}_{2\leftrightarrow 2} + \mathrm{Coll}_{3\leftrightarrow 1}$, defined by

$$\mathrm{Coll}_s = \frac{3\pi A^3}{2(hc)^3}\int_{\Omega_s}\frac{S_{\omega_k,\omega_1,\omega_2,\omega_3}}{\omega_k^\alpha\omega_1^\alpha\omega_2^\alpha\omega_3^\alpha}(\omega_k^\alpha + s\omega_1^\alpha - \omega_2^\alpha - \omega_3^\alpha) \times\left[1 + s\left(\frac{\omega_1}{\omega_k}\right)^\beta - \left(\frac{\omega_2}{\omega_k}\right)^\beta - \left(\frac{\omega_3}{\omega_k}\right)^\beta\right]d\omega_2 d\omega_3.$$

Here $\beta = 3\alpha - 2$. For $\mathrm{Coll}_{2\leftrightarrow 2}$ one has that $s \equiv 1$ and the integration domain is over $\Omega_+ = \{0 \le \omega_2 \le \omega_k, \omega_k - \omega_2 \le \omega_3 \le \omega_k\}$ and $\omega_1 = \omega_2 + \omega_3 - \omega_k$, while for $\mathrm{Coll}_{3\leftrightarrow 1}$ one has that $s \equiv -1$ and the integration is over $\Omega_- = \{0 \le \omega_2 \le \omega_k, 0 \le \omega_3 \le \omega_k - \omega_2\}$, with $\omega_1 = \omega_k - \omega_2 - \omega_3$. The collisional terms scale as $\mathrm{Coll}_{2\leftrightarrow 2} = C_+(\alpha)\omega_k^{1-3\alpha}$ and $\mathrm{Coll}_{3\leftrightarrow 1} = C_-(\alpha)\omega_k^{1-3\alpha}$. The coefficients $C_\pm(\alpha)$ are pure real functions depending only on α. Although, the explicit form of the scattering matrix $S_{\omega_1,\omega_2,\omega_3,\omega_4}$ is not simple, its value can be bounded in both domains $\Omega_\pm$ and the collision term converges for suitable values of $\alpha \in (0.5, 1.2)$ validating the locality condition. Both coefficients vanish with double degeneracy at $\alpha = 1$ indicating that the KZ spectrum: $n_k^{\mathrm{KZ}} \sim \frac{1}{\omega_k} \sim \frac{1}{k^2}$ coincides with the Rayleigh-Jeans solution, Eq. (5). It means, in fact, that the energy flux is zero. The double degeneracy at $\alpha = 1$ reveals the existence of a logarithmic correction, similarly to the case of the nonlinear Schrödinger equation (NLS) in two dimensions [8]. As stated in Ref. [21] the logarithmic correction produces a divergent result for NLS. In our case, it is possible to show that all integrals are finite, indicating a finite energy flux [22]. Thus one has

$$n_k^{\mathrm{KZ}} = C\frac{hP^{1/3}\rho^{2/3}}{[12(1-\sigma^2)]^{2/3}}\frac{\ln^{1/3}(k_*/k)}{k^2}, \tag{6}$$

where P is the energy flux. C and k_* are real numbers.

For $\alpha = 0$ and $3\alpha - 2 = 0$ the collisional part $\mathrm{Coll}_{2\leftrightarrow 2}$ also vanishes. This solution corresponds to the wave action equipartition ($\alpha = 0$) with a second KZ spectrum $n_k \sim 1/\omega_k^{2/3}$ related to wave action inverse cascade. However, this spectrum does not vanish the second part of the collision term $\mathrm{Coll}_{3\leftrightarrow 1}$, in agreement with the nonconservation of the wave action mentioned above. In conclusion there exist only a single cascade: the energy cascade (6).

Numerical simulation.—Numerical simulations of the full nonlinear system of Eqs. (1) and (2) are first performed to validate the formation of the equilibrium spectrum Eq. (5). In all the presented results $c = 1$ and $h = 1/2$ so that the linear plate size is the only parameter of the numerics. We have implemented a pseudospectral scheme using FFT routines [23], with periodic boundary conditions: the linear part of the dynamics is calculated exactly in Fourier space: $\zeta_{\boldsymbol{k}}(t + \Delta t) = \zeta_{\boldsymbol{k}}(t)\cos(\omega_k\Delta t) + \frac{\dot\zeta_{\boldsymbol{k}}(t)}{\omega_k}\times\sin(\omega_k\Delta t)$. The nonlinear terms in (1) and (2) are first computed in real space and the integration in time is then performed in Fourier space using an Adams-Bashford scheme. It interpolates the nonlinear term of (1) as a polynomial function of time (of order one in the present calculations). Energy is conserved within a 1/100 relative error. As initial conditions, we have taken: $\zeta_k = \zeta_0 e^{-k^2/k_0^2}e^{i\varphi_k}$ with φ_k a random phase, and a zero velocity field $\dot\zeta_k = 0$. As time evolves, the random waves oscillate with a disorganized behavior, as shown in Fig. 1. After a long time the system builds up an equilibrium distribution in agreement with the Rayleigh-Jeans $n_k \sim T/k^2$ spectrum. That is, for the plate deflection $\langle|\zeta_k|^2\rangle = X_k^2 n_k = \frac{n_k}{\rho\omega_k} = \frac{T}{\rho h^2 c^2 k^4}$ as shown in Fig. 2.

Nonequilibrium distributions can also be observed numerically. One requires to input energy and pump wave action at low wave numbers ($k < k_{\mathrm{in}}$) and to dissipate energy at large ones ($k > k_{\mathrm{out}}$) defining a window of transparency $k_{\mathrm{in}} < k < k_{\mathrm{out}}$. This artifact is implemented by adding a term $(F_{\boldsymbol{k}} - \gamma_k\dot\zeta_{\boldsymbol{k}})$ to the plate Eq. (1). Following [14] the forcing term $F_{\boldsymbol{k}}$ is a nonzero random force for $k < k_{\mathrm{in}}$, and γ_k is a fictitious linear damping for short length scales. Figure 3 shows a good agreement with the predicted KZ spectrum Eq. (6) with an exponent for the logarithmic correction 1/3 (inset of Fig. 3).

Conclusions.—We have successfully applied weak turbulence theory for the new case of elastic thin plates. The results allow for an analogy between an important property of fluid dynamics and the mechanics of elastic plates. Numerical simulations exhibit both the convergence towards statistical equilibrium for a free system and an energy cascade when forcing and dissipation are introduced, as predicted by the weak turbulence analysis. An important consequence is the nonexistence of an inverse

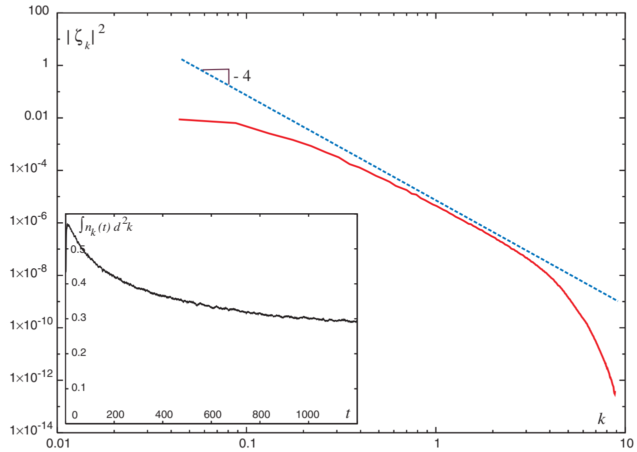

FIG. 2 (color online). Numerical simulation for a 512 square plate using 1024^2 modes with a mesh size $dx = 1/2$. The initial condition is with $k_0 = 1$ and $\zeta_0 = 0.02$. We plot the power spectrum of the mean deflection $\langle|\zeta_k|^2\rangle$ versus wave number k after 1200 time units. The line plots the Rayleigh-Jeans power law $7 \times 10^{-6}/k^4$ which gives $T \approx 2 \times 10^{-6}$ in agreement with the equipartition of the initial energy. The inset plots the evolution of the wave action with time.

cascade $n_k \sim 1/k^{4/3}$, as usually found for four wave interaction systems such as gravity waves or NLS. The results presented here suggest also a new experimental way of studying weak turbulence through the analysis of the waves produced by the plate oscillations [24].

For large deformations the elastic plate equations are still valid, but stretching cannot be longer treated as a weak perturbation and a "wave breaking" phenomenon is expected: energy focuses into localized structures as ridges [25] and conical surfaces (named d-cones) [26]. Amazingly, a regime dominated by ridges shows a power spectrum $|\zeta_k|^2 \sim 1/k^4$ similar to the weak turbulence spectrum derived here. On the other hand for d-cone-dominated

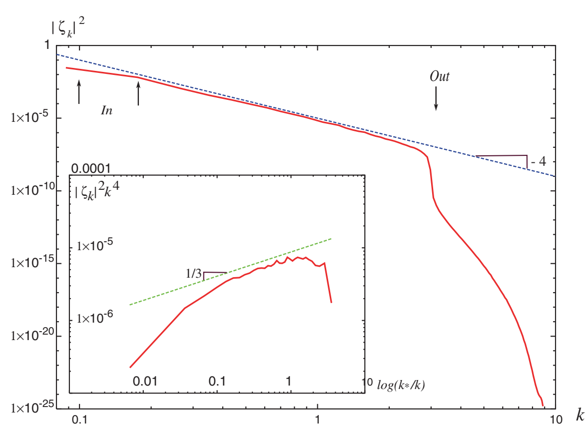

FIG. 3 (color online). Average power spectrum $\langle|\zeta_k|^2\rangle$ for the energy cascade. The injection scale is $k_{\rm in} \in (0.1, 0.25)$ while the dissipation is at $k_{\rm out} = 3$. The line plots the power law $1/k^4$. Inset plots $k^4\langle|\zeta_k|^2\rangle$ vs $\log(k_*/k)$ in logarithmic scale with $k_* = k_{\rm out}$. The straight line corresponds to $z = 1/3$.

regimes, as seemingly observed in [27], the expected spectrum should follow $|\zeta_k|^2 \sim 1/k^6$.

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