Cours 2 : Conduction stationnaire

Transferts de chaleur

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- *10h cours (5 séances)*
- *18h TD (9 séances)*

- *Conduction*
- *Convection*
- *Rayonnement*
- *Transferts de chaleur « multimodes »*

Documents de cours sur:

https://moodle.insa-lyon.fr Mes cours : Transferts de chaleur

https://moodle.insa-lyon.fr/course/view.php?id=4885

Conduction – notions générales

- **Diffusion**
  - dans la direction du gradient de température
  - milieu solide et fluide

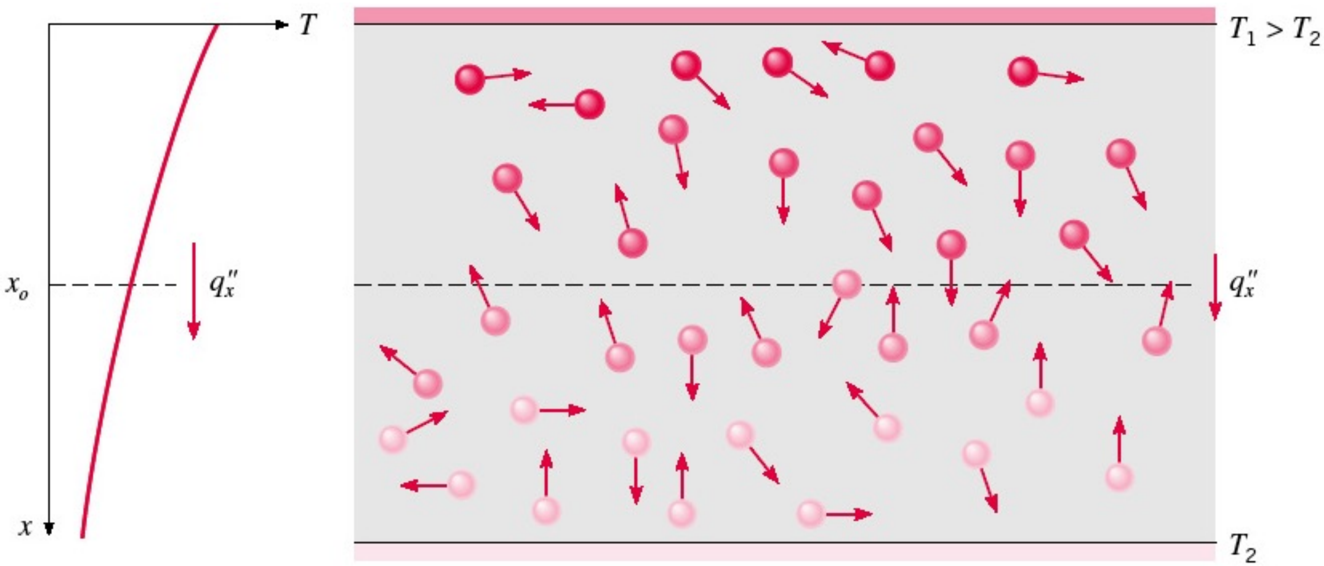
- **Problèmes**

Types de problèmes dans la conduction de la chaleur

Problème	Conditions aux limites	Caractéristiques du volume	Distribution de la température
Direct			
Simulation	Données	Données	Demandée
Inverse			
Identification	Données	Demandées	Données
Contrôle	Demandées	Données	Donnée

Conduction – notions générales

- **Diffusion**

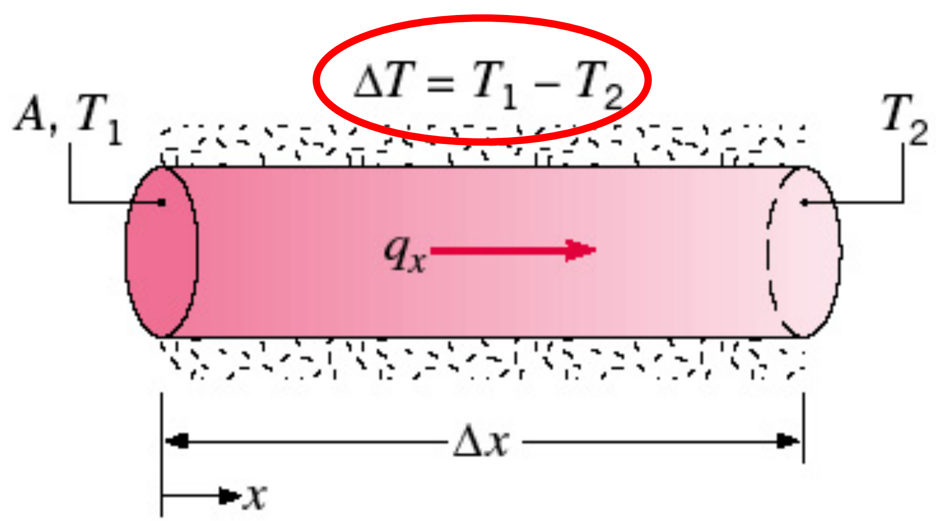


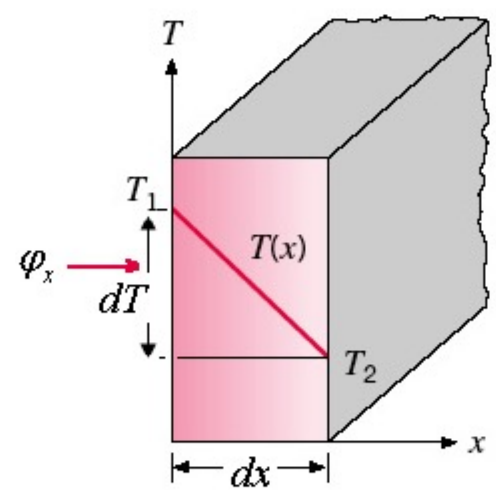
Loi constitutive : Fourier

- **loi phénoménologique**
- **conductivité thermique**
  - propriété de matériel
  - mesurée expérimentalement

$$\dot{Q}_x \equiv q_x = -\lambda A \frac{dT}{dx}$$

$$\varphi = -\lambda \frac{dT}{dx}$$





Conduction – notions générales
Loi constitutive : Fourier

- **Loi de Fourier**

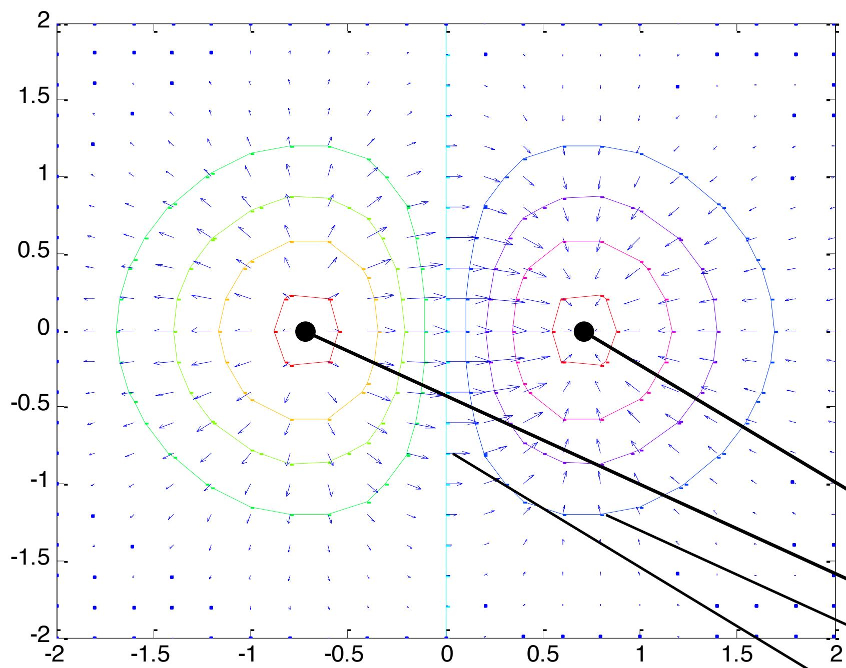
$$\boldsymbol{\varphi} = -\lambda \frac{\partial T}{\partial n}\mathbf{n} \qquad \boldsymbol{\varphi} = -\lambda \mathbf{grad}\, T$$

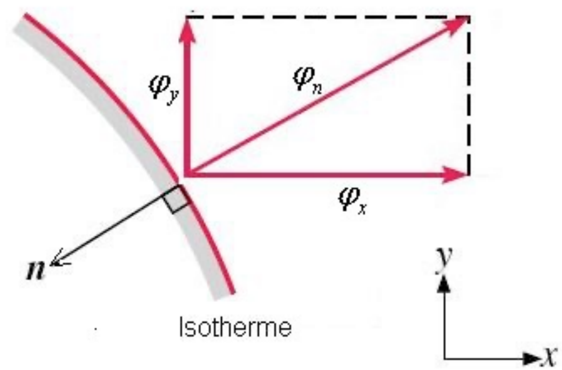
$$\frac{\partial T}{\partial n}\mathbf{n} \equiv \mathbf{grad}\, T \equiv \nabla T$$

Type de système	Coordonnées	Gradient
Une seule dimension	$\mathbf{n}$	
Trois dimensions		
Cartésiennes	$\mathbf{x}, \mathbf{y}, \mathbf{z}$	$\mathbf{grad}(T) = \frac{\partial T}{\partial x}\mathbf{x} + \frac{\partial T}{\partial y}\mathbf{y} + \frac{\partial T}{\partial z}\mathbf{z}$
Cylindriques	$\mathbf{r}, \boldsymbol{\theta}, \mathbf{z}$	$\mathbf{grad}(T) = \frac{\partial T}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{\theta} + \frac{\partial T}{\partial z}\mathbf{z}$
Sphériques	$\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}$	$\mathbf{grad}(T) = \frac{\partial T}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \omega}\boldsymbol{\omega}$

Conduction – notions générales
Loi constitutive : Fourier

- **Distribution de la température (scalaire) et champs de flux (vecteur)**





Puits de chaleur ($div\boldsymbol{\varphi} < 0$)

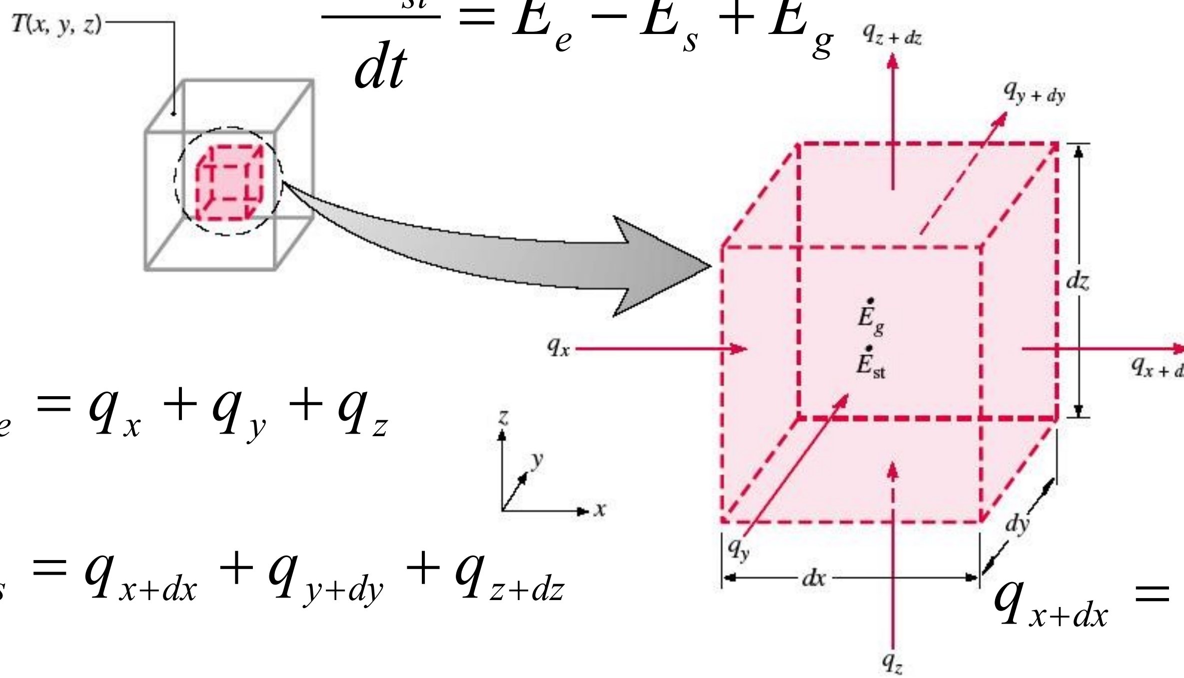
Source de chaleur ($div\boldsymbol{\varphi} > 0$)

Isotherme

Densité de flux de chaleur

- **Bilan d'énergie pour un domaine de contrôle**

$$\frac{dE_{st}}{dt} = \dot{E}_e - \dot{E}_s + \dot{E}_g$$



$$\dot{E}_e = q_x + q_y + q_z$$

$$\dot{E}_s = q_{x+dx} + q_{y+dy} + q_{z+dz}$$

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$\underbrace{-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz}_{\dot{E}_e - \dot{E}_s} + \underbrace{p\, dx\, dy\, dz}_{\dot{E}_g} = \underbrace{\rho c\, dx\, dy\, dz \frac{\partial T}{\partial t}}_{\frac{dE_{st}}{dt}}$$

Equation de la chaleur

- **Forme différentielle du bilan d'énergie**

$$\varphi_x \equiv \frac{q_x}{dy\,dz} \qquad \varphi_y \equiv \frac{q_y}{dx\,dz} \qquad \varphi_z \equiv \frac{q_z}{dx\,dy}$$

$$\underbrace{-\frac{\partial \varphi_x}{\partial x} - \frac{\partial \varphi_y}{\partial y} - \frac{\partial \varphi_z}{\partial z}}_{\dot{E}_e - \dot{E}_s} + \underbrace{p}_{\dot{E}_g} = \underbrace{\rho c \frac{\partial T}{\partial t}}_{\frac{dE_{st}}{dt}}$$

Unité de volume: $\dot{E}_e - \dot{E}_s \qquad \dot{E}_g \qquad \frac{dE_{st}}{dt}$

$$-div\boldsymbol{\varphi} + p = \rho c \frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} + div\left(\frac{\boldsymbol{\varphi}}{\rho c}\right) = \frac{p}{\rho c}$$

Equation de la chaleur

- **Equation générale de la chaleur**

$$\frac{\partial T}{\partial t} = -div\left(\frac{\boldsymbol{\varphi}}{\rho c}\right) + \frac{p}{\rho c}$$ équation de continuité (fondamentale)

$$\boldsymbol{\varphi} = -\lambda \mathbf{grad} T$$ loi de Fourier (empirique)

$$\rho c \frac{\partial T}{\partial t} = div(\lambda\, \mathbf{grad} T) + p$$ équation de la chaleur

$$\frac{\partial T}{\partial t} = div(\alpha\, \mathbf{grad} T) + \frac{p}{\rho c}$$

Equation de la chaleur

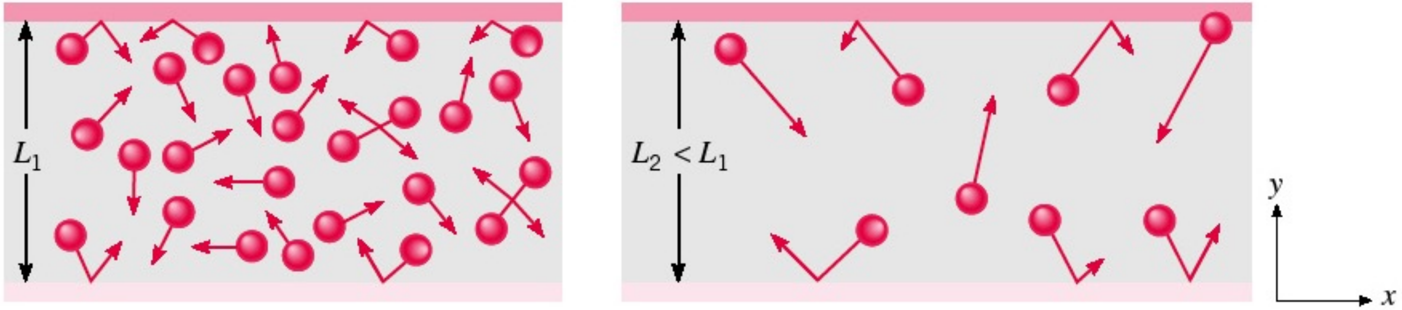
- **Note : effets à micro-échelle**

$$\boldsymbol{\varphi} = -\lambda\ \mathbf{grad}\,T$$

loi de Fourier (**empirique**) :

- effets de bord à l'échelle micro- et nano-metrique

- matériaux nano-structurés



Conduction – notions générales
Equation de la chaleur

- **Schéma du raisonnement**

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
T \arrow[d, "\mathbf{grad}", "\text{Connexion}"'] & p - \rho c \dfrac{\partial T}{\partial t} \\
\nabla T \arrow[r, "\text{Loi constitutive}", "-\lambda"'] & \boldsymbol{\varphi} \arrow[u, "div"', "\text{Bilan}"]
\end{tikzcd}
$$

$$\boldsymbol{\varphi} = -\lambda\, \mathbf{grad} T$$

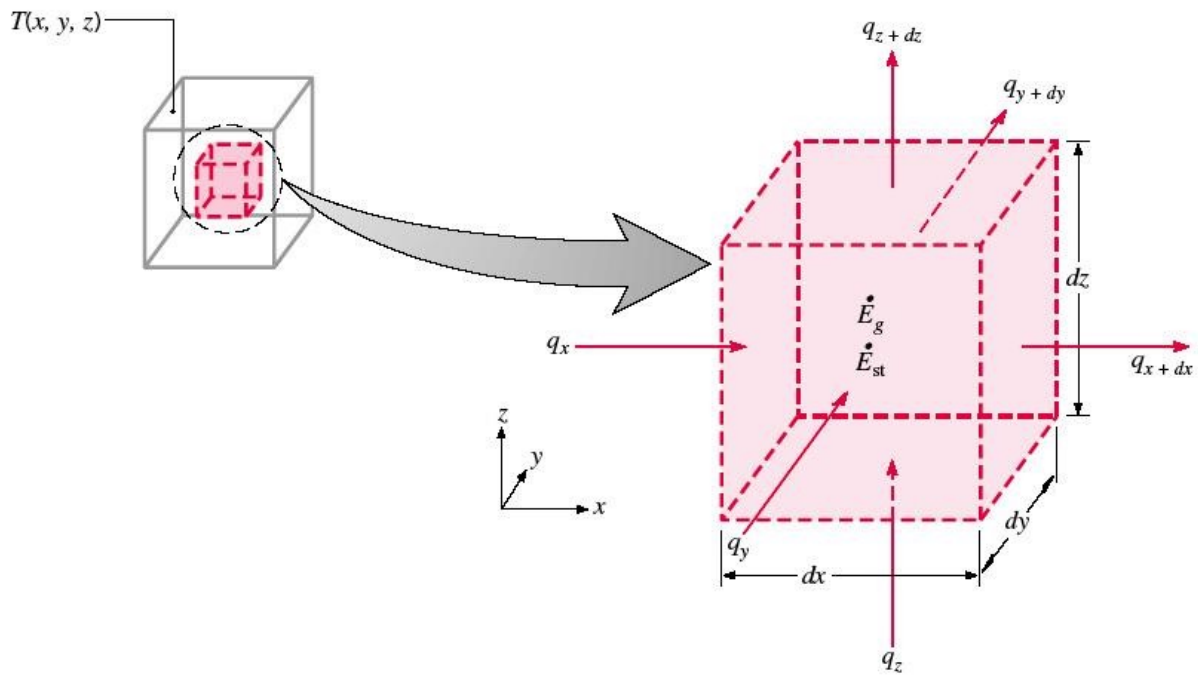
$$div(\boldsymbol{\varphi}) = p - \rho c \frac{\partial T}{\partial t}$$

T	- potentiel
∇T	- différence de potentiel
φ	- flux dus aux différences de potentiel
f	- flux externes

Conduction – notions générales
Equation de la chaleur – formes particulaires

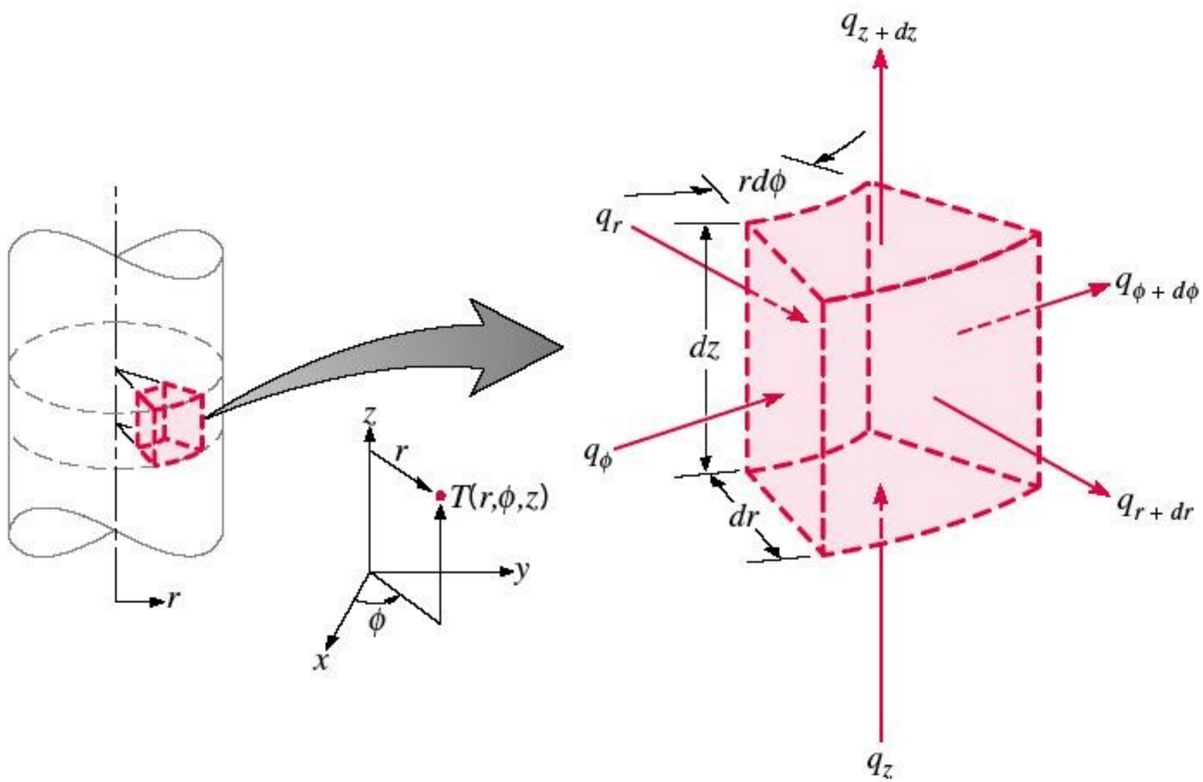
- **Coordonnées cartésiennes**

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+p=\rho\, c\,\frac{\partial T}{\partial t}$$



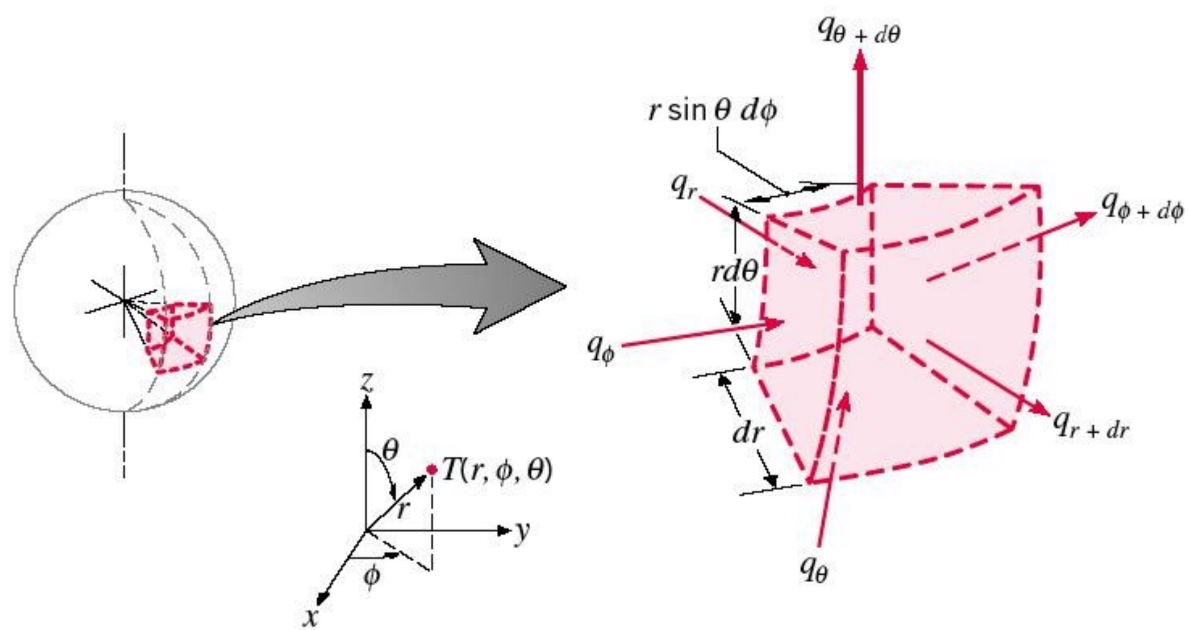
- **Coordonnées cylindriques**

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\lambda\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+p=\rho c\frac{\partial T}{\partial t}$$



- **Coordonnées sphériques**

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\lambda\frac{\partial T}{\partial\phi}\right)$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\lambda\sin\theta\frac{\partial T}{\partial\theta}\right)+p=\rho c\frac{\partial T}{\partial t}$$



- **Matériel homogène et isotrope** $\lambda = const.$

$$\rho c \frac{\partial T}{\partial t} = \lambda \, div(\mathbf{grad} T) + p$$

$$div(\mathbf{grad} T) = \Delta T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Laplacien

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = div(\mathbf{grad} T) + \frac{p}{\lambda}$$

$$\alpha = \frac{\lambda}{\rho c}$$

Diffusivité

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \Delta T + \frac{p}{\lambda}$$

- **Matériel homogène isotrope** $\lambda = const.$

en régime stationnaire $\frac{\partial T}{\partial t} = 0$

équation de Poisson :

$$\lambda\, div(\mathbf{grad}T) + p = 0$$

$$\lambda\, \Delta T + p = 0$$

- **Matériel homogène isotrope** $\lambda = const.$

en régime stationnaire $\frac{\partial T}{\partial t} = 0$

sans sources internes $p = 0$

équation de Laplace :

$$div(\mathbf{grad}T) = 0$$

$$\Delta T = 0$$

Conditions initiales et aux limites

$$\rho c \frac{\partial T}{\partial t} = \underbrace{div(\lambda \mathbf{grad} T)}_{} + p$$

pour chaque dimension

2 conditions aux limites spatiales

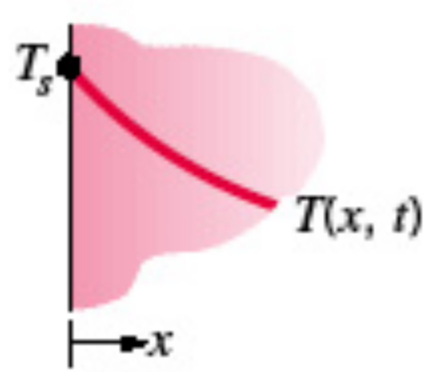
distribution spatiale de la température au moment initial

- **Condition initiale**

$$T\Big|_{t=0} \equiv T_0 = f(x, y, z, 0)$$

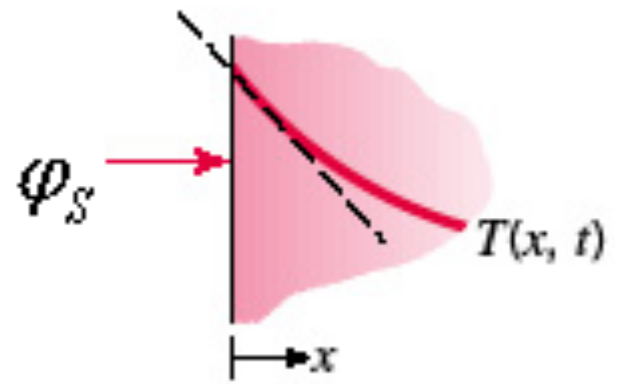
- **Condition de Dirichelt**

$$T(0,t) = T_S$$



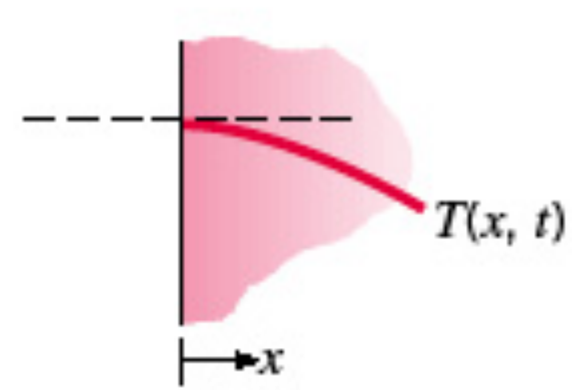
- **Condition de Neumann**
  - densité de flux imposée

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = \varphi_S$$



  - surface adiabatique ou de symétrie

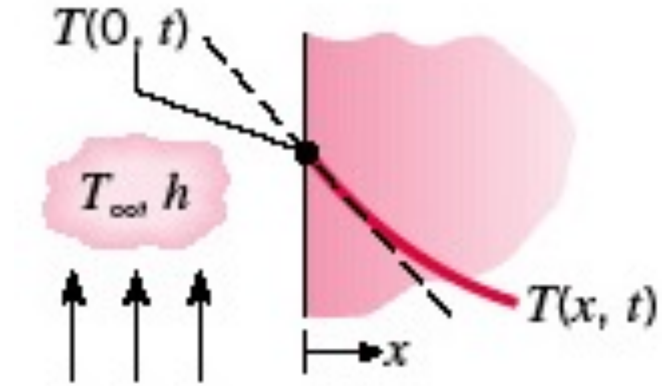
$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$



Conduction – notions générales
Conditions initiales et aux limites

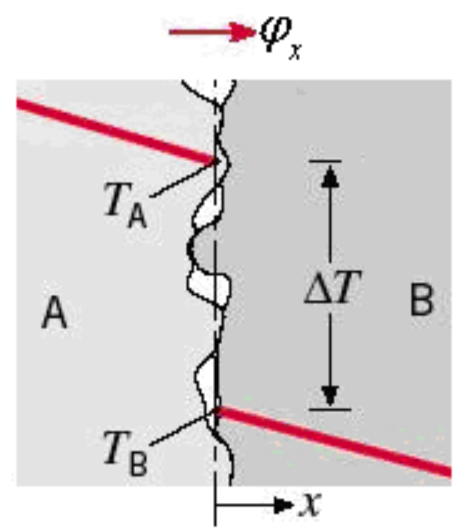
- **Condition de Fourier**

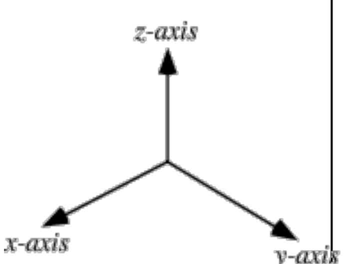
$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = h_c[T_\infty - T(0,t)]$$



- **Surfaces en contacte**

$$\lambda_A \frac{\partial T}{\partial x}\bigg|_{x=0^-} = \lambda_B \frac{\partial T}{\partial x}\bigg|_{x=0^+}$$





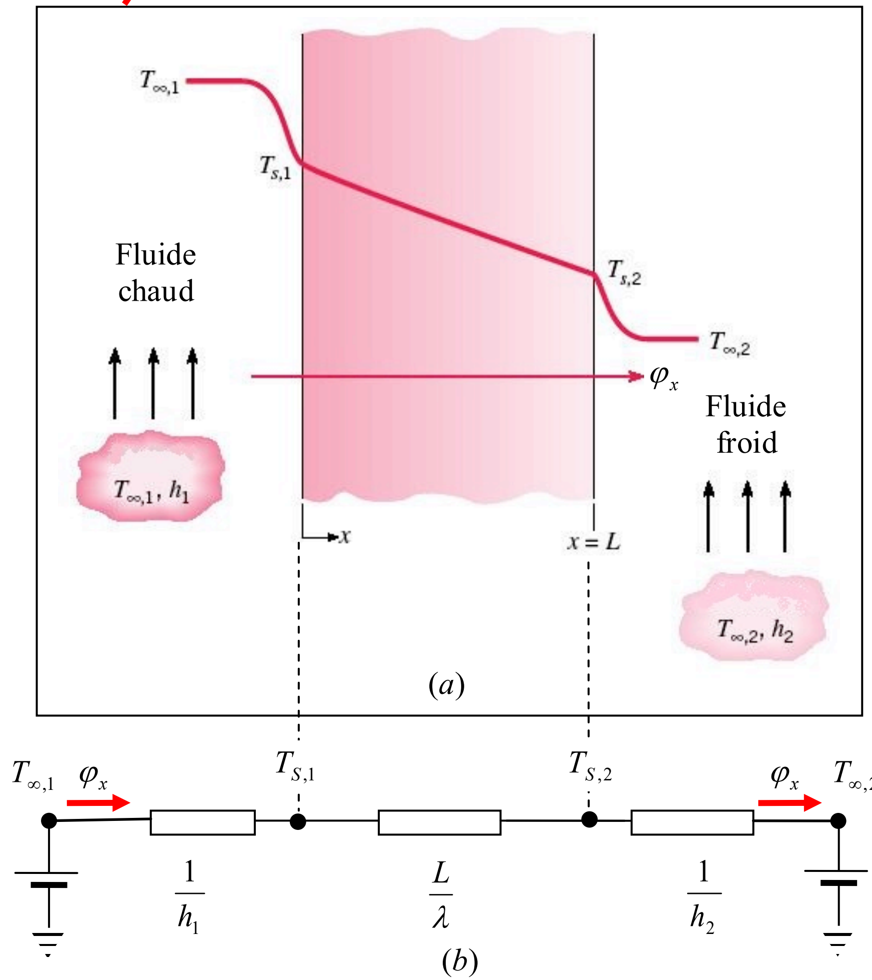
$$\cancel{\rho c \frac{\partial T}{\partial t}} = div(\lambda \mathbf{grad} T) + \cancel{p}$$

$$div(\lambda \ \mathbf{grad} T) = 0$$

- **Mur plan**

$$\begin{cases} \dfrac{d}{dx}\left(\lambda \dfrac{dT}{dx}\right) = 0 \\ T(0) = T_{s1} \\ T(L) = T_{s2} \end{cases}$$

$$T(x) = C_1 x + C_2$$



- **Conditions aux limites**

$$T(0) = T_{S1} \qquad T(L) = T_{S2}$$

  - ➔ constantes d'intégration

$$C_2 = T_{S1} \qquad C_1 = \frac{T_{S2} - T_{S1}}{L}$$

  - ➔ solution particulière

$$T(x) = \frac{T_{S2} - T_{S1}}{L} x + T_{S1}$$

- **loi Fourier ➔**

$$\varphi_x = -\frac{\lambda}{L}(T_{S2} - T_{S1}) = \frac{\lambda}{L}(T_{S1} - T_{S2})$$

- **flux**

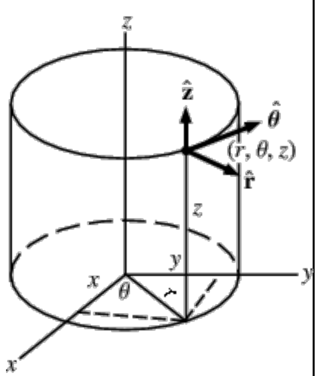
$$q_x = A\varphi = \lambda \frac{A}{L}(T_{S1} - T_{S2})$$

Conduction unidimensionnelle régime stationnaire

Conduction sans sources internes

Conduction

- Conduction - général
  - Loi Fourier
  - Equation chaleur
  - Formes particulaires
  - Conditions limites
- Conduction 1D
  - Sans sources
  - Résistance therm.
  - Avec sources
  - Circuits thermiques

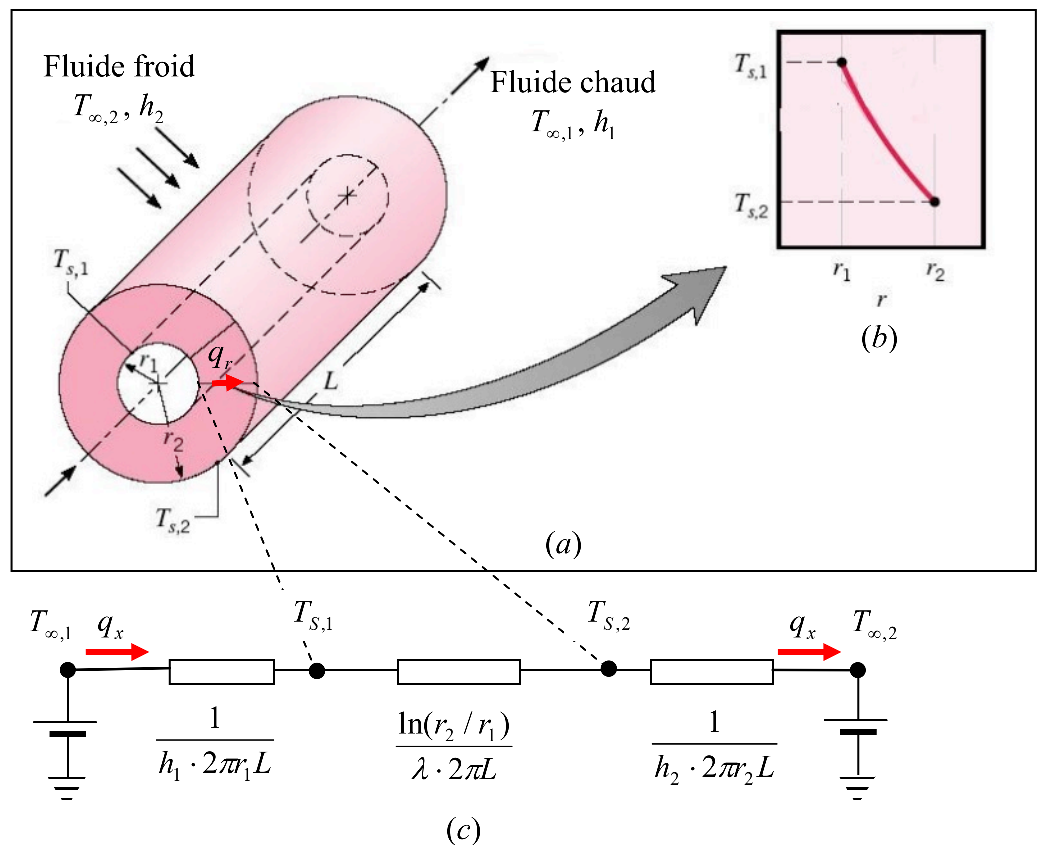


- **Cylindre**

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r\frac{\partial T}{\partial r}\right)+\cancel{\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\lambda\frac{\partial T}{\partial r}\right)}+\cancel{\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)}+\cancel{p}=\cancel{\rho c\frac{\partial T}{\partial t}}$$

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left(\lambda r\dfrac{dT}{dr}\right)=0 \\ T(r_1)=T_{s1} \\ T(r_2)=T_{s2} \end{cases}$$

$$T(r)=C_1\ln r+C_2$$



- **Conditions aux limites** $T(r_1) = T_{S1}$ $T(r_2) = T_{S2}$
  - ➔ constantes d'intégration

$$C_1 = \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \qquad C_2 = \frac{T_2 \ln r_1 - T_1 \ln r_2}{\ln \frac{r_1}{r_2}}$$

  - ➔ solution particulière

$$T(r) = T_{S1} - \frac{T_{S1} - T_{S2}}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_1} = T_{S2} + \frac{T_{S1} - T_{S2}}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_2}$$

- **flux** $q_r = \lambda \dfrac{2\pi L}{\ln(r_2 / r_1)} (T_{S1} - T_{S2})$

Conduction unidimensionnelle régime stationnaire

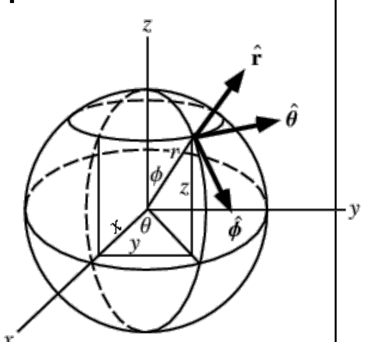
Conduction sans sources internes

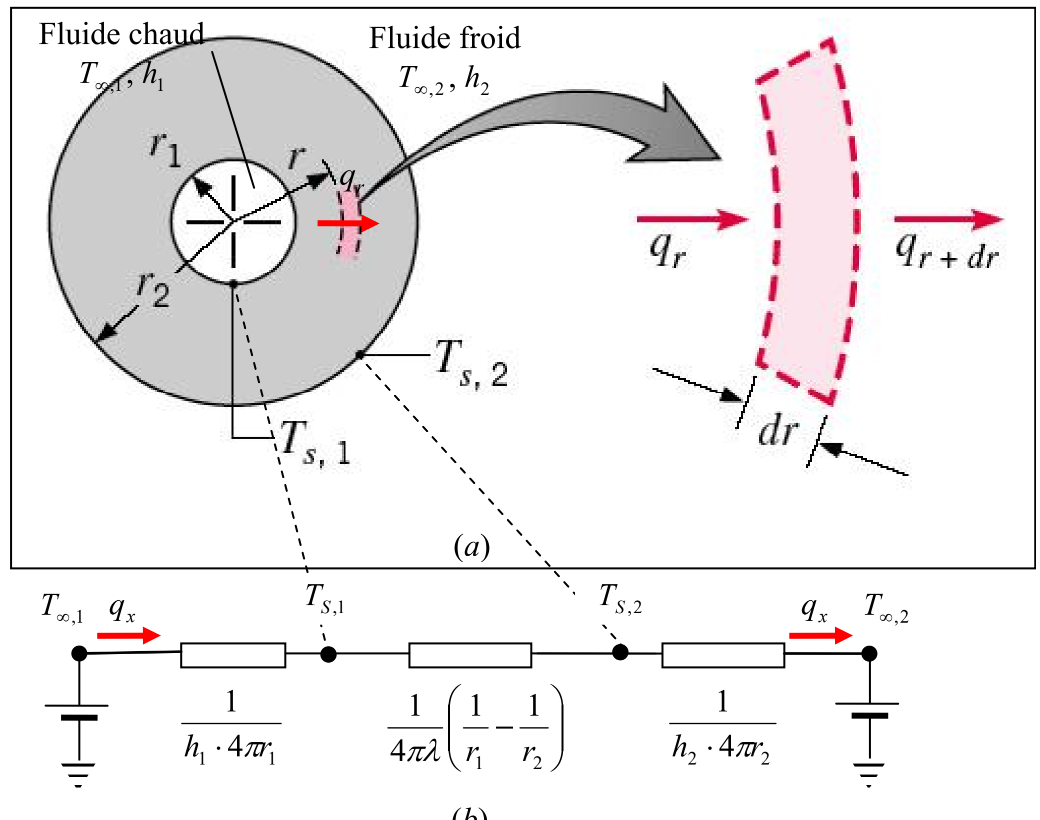
- **Sphère**

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\cancel{\frac{\partial}{\partial \phi}}\left(\lambda\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\cancel{\frac{\partial}{\partial \theta}}\left(\lambda\sin\theta\frac{\partial T}{\partial z}\right)+\cancel{p}=\rho c\cancel{\frac{\partial T}{\partial t}}$$

$$\begin{cases}\dfrac{1}{r^2}\dfrac{d}{dr}\left(\lambda r^2\dfrac{dT}{dr}\right)=0\\ T(r_1)=T_{s1}\\ T(r_2)=T_{s2}\end{cases}$$

$$T(r)=C_1/r+C_2$$





- **Conditions aux limites** $T(r_1) = T_{S1}$ $T(r_2) = T_{S2}$
  - ➔ constantes d'intégration

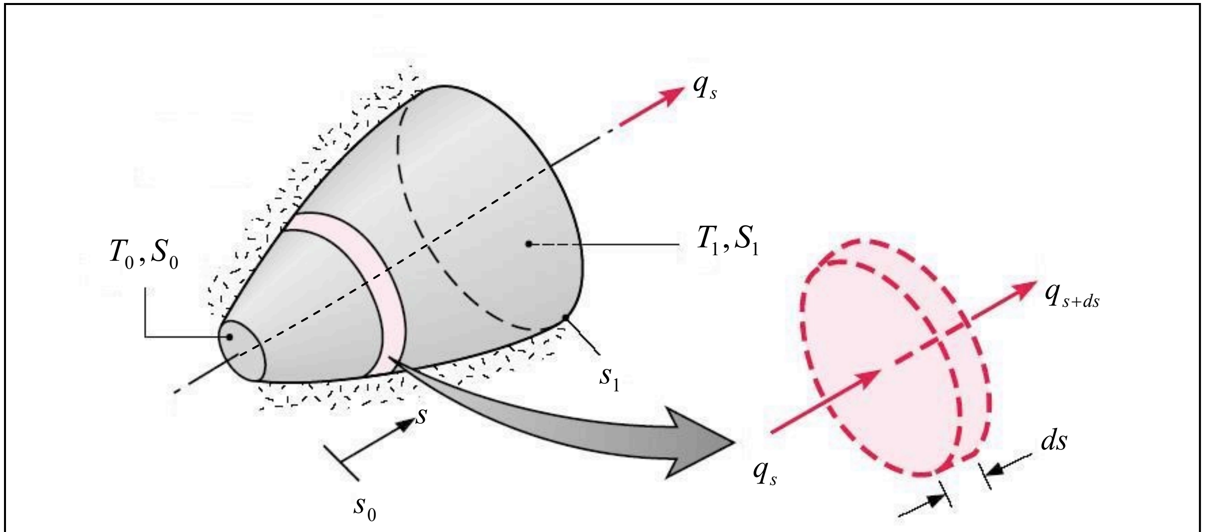
$$C_1 = \frac{r_1 r_2 (T_{S1} - T_{S2})}{(r_2 - r_1)} \qquad C_2 = \frac{T_{S2} r_2 - T_{S1} r_1}{r_2 - r_1}$$

  - ➔ solution particulière

$$T(r) = T_{S1} + \frac{r_1 r_2 (T_1 - T_2)}{r_2 - r_1} \left(\frac{1}{r} - \frac{1}{r_1} \right)$$

- **flux**

$$q_r = \lambda \frac{4\pi}{\frac{1}{r_1} - \frac{1}{r_2}} (T_{S1} - T_{S2})$$



conduction stationnaire sans sources internes de chaleur

- **Loi de Fourier dans une section**

$$q = S(s)\,\varphi = -S(s)\,\lambda(s)\left.\frac{dT}{ds}\right|_s$$

q est constant !

– Séparation des variables

$$q\frac{ds}{S(s)\,\lambda(s)} = -dT$$

$$q\int_{s_0}^{s}\frac{ds}{S(s)\lambda(s)} = -\int_{T_0}^{T} dT \quad \Leftrightarrow \quad qR = T_0 - T_1$$

$$R = \int_{s_0}^{s}\frac{ds}{S(s)\lambda(s)} \qquad R_{cd} \equiv \frac{T_0 - T_1}{q}$$

$$R_{cd} = \frac{1}{\lambda}\frac{L}{A}$$

mur plan

$$R_{cd} = \frac{1}{\lambda}\frac{\ln(r_2/r_1)}{2\pi L}$$

cylindre

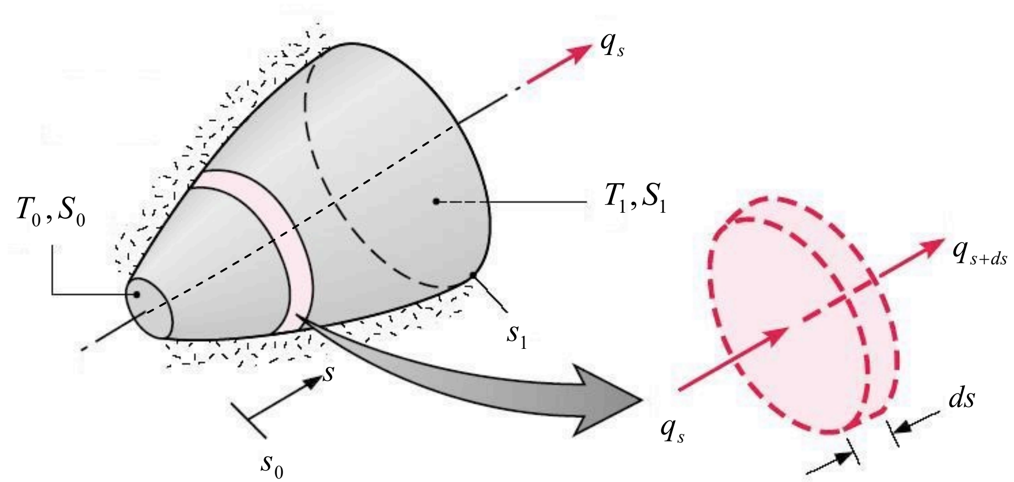
$$R_{cd} = \frac{1}{\lambda}\frac{1}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

sphère

Conduction unidimensionnelle régime stationnaire

Avec sources internes

- **Pour un tube de courent**



s -abscisse curviligne sur la ligne de courant
S -surface orthogonale aux lignes de courant

$$q_s - q_{s+ds} + p\,S\,ds = 0$$ Conduction stationnaire

$$dq = p\,S\,ds$$

$$\int_{q_0}^{q} dq = \int_{s_0}^{s} p\,S\,ds$$

$$\boxed{q(s) = \int_{s_0}^{s} pS\,ds + q_0}$$

- **Pour un tube de courent**

$$q(s) = -\lambda S \frac{dT}{ds}$$

$$q(s) = \int_{s_0}^{s} pS\, ds + q_0$$

$$f_1 = \int_{s_0}^{s_1} pS\, ds$$

$$\lambda S \frac{dT}{ds} = -\int_{s_0}^{s} p\, S\, ds - q_0$$

$$q_1 = f + q_0$$

$$dT = \frac{1}{\lambda S}\left[-\int_{s_0}^{s} p\, S\, ds'\right] ds - q_0 \frac{ds}{\lambda S}$$

$$T_1 = \int_{s_0}^{s_1} \frac{1}{\lambda S}\left[\int_{s_0}^{s} - p\, S\, ds'\right] ds - q_0 \int_{s_0}^{s_1} \frac{ds}{\lambda S} + T_0$$

Conduction unidimensionnelle régime stationnaire

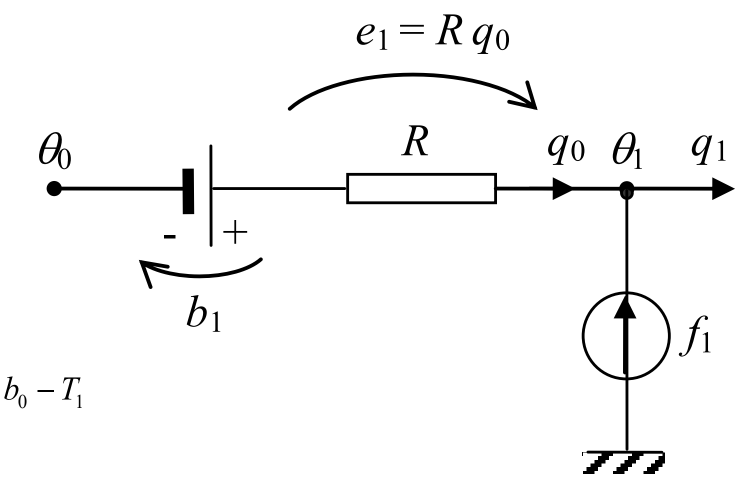
Avec sources internes

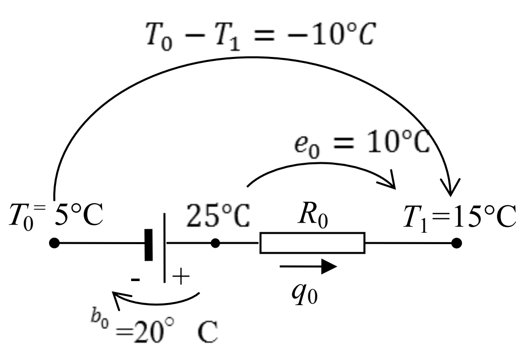
$$dr = \frac{ds}{\lambda S} \qquad R = \int_{s_0}^{s_1} \frac{ds}{\lambda S}$$

$$T_1 = \underbrace{-r_1 \int_{s_0}^{s_1} p\,S\,ds + \int_{s_0}^{s_1} p\,S\,r\,ds}_{b_1} \underbrace{- R\,q_0}_{e_1} + T_0$$

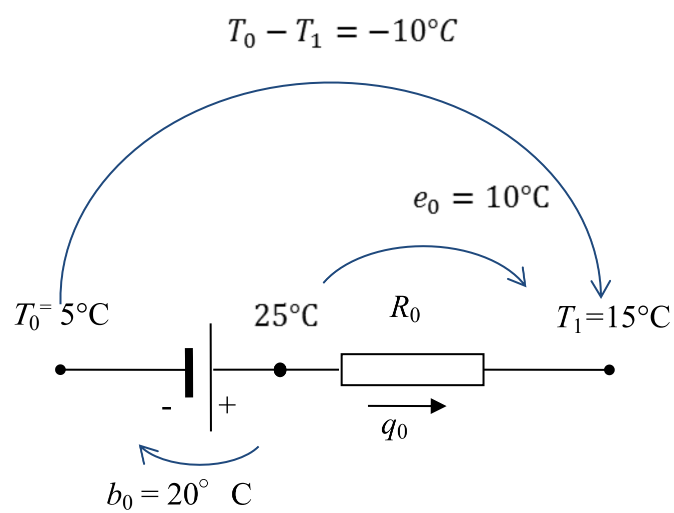
$$f_1 = \int_{s_0}^{s_1} pS\,ds \qquad b_1 = -r_1 \int_{s_0}^{s_1} pS\,ds + \int_{s_0}^{s_1} pSr\,ds$$

$$\begin{cases} q_1 = q_0 + f_1 \\ \theta_1 = b_1 - e_1 + \theta_0 \end{cases}$$





$$e_0 = T_0 + b_0 - T_1$$



$$e_0 = T_0 + b_0 - T_1$$

- **Resistance de convection**

$$R_{cv} \equiv \frac{T_S - T_\infty}{q} = \frac{1}{h_{cv} A}$$

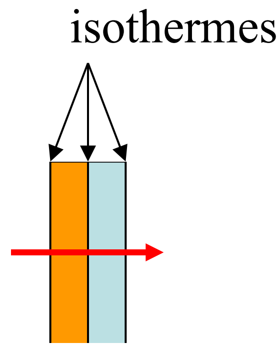
- **et de rayonnement**

$$R_r \equiv \frac{T_S - T_{me}}{q} = \frac{1}{h_r A}$$

Conduction
Conduction - général
Loi Fourier
Equation chaleur
Formes particulaires
Conditions limites
Conduction 1D
Sans sources
Résistance therm.
Avec sources
Circuits thermiques

- **Circuits thermiques**
  - série

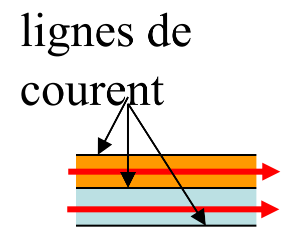
$$R_{ser} = \sum_{i=1}^{n} R_i$$



$$\frac{1}{K_{ser}} = \sum_{i=1}^{n} \frac{1}{K_i}$$

parallèle

$$\frac{1}{R_{par}} = \sum_{i=1}^{n} \frac{1}{R_i}$$



$$K_{par} = \sum_{i=1}^{n} K_i$$

- **Résistance convection || rayonnement**

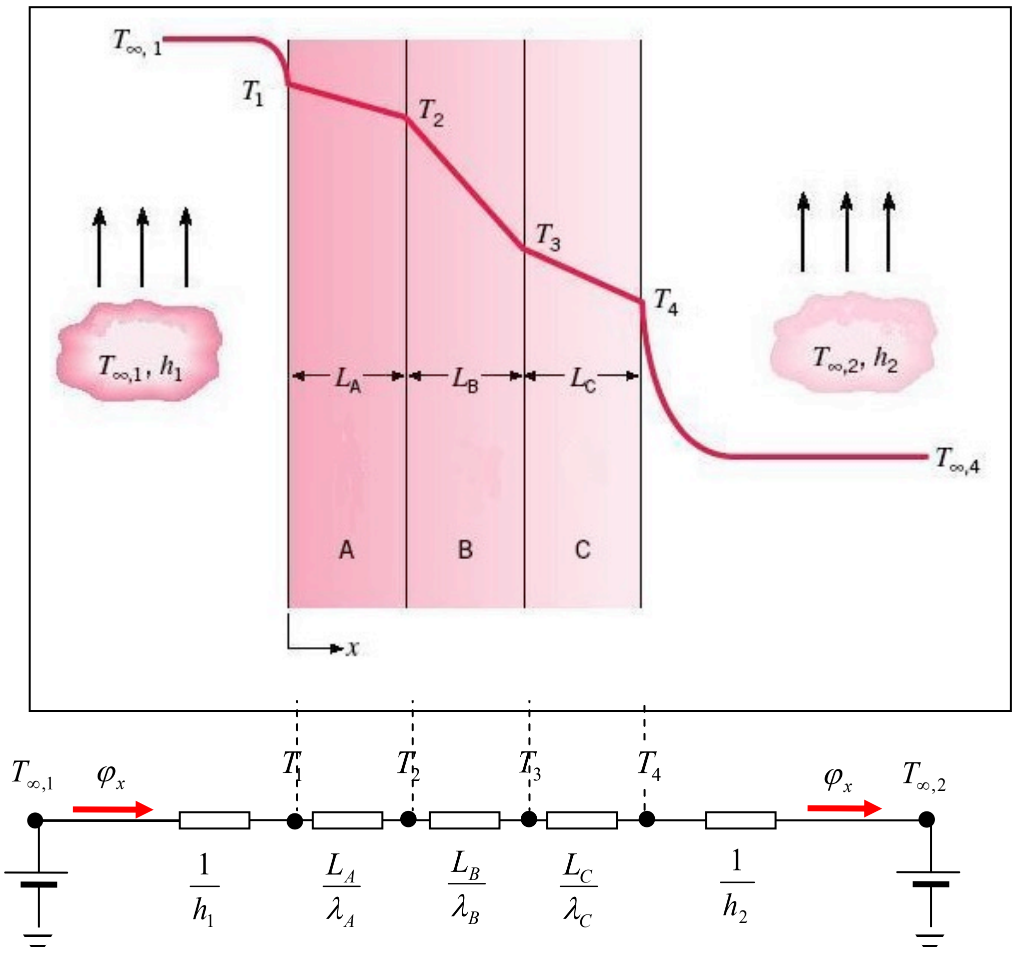
$$\frac{1}{R_s} = \frac{1}{R_{cv}} + \frac{1}{R_r} = (h_{cv} + h_r)A = hA$$

- **Mur plan : conv. + cond. + conv.**

$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{\lambda A} + \frac{1}{h_2 A}$$

Conduction unidimensionnelle régime stationnaire
Circuits thermiques

- **Mur multicouche**



- **Mur multicouche**

$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{\lambda_A A} + \frac{L_B}{\lambda_B A} + \frac{L_C}{\lambda_C A} + \frac{1}{h_2 A}$$

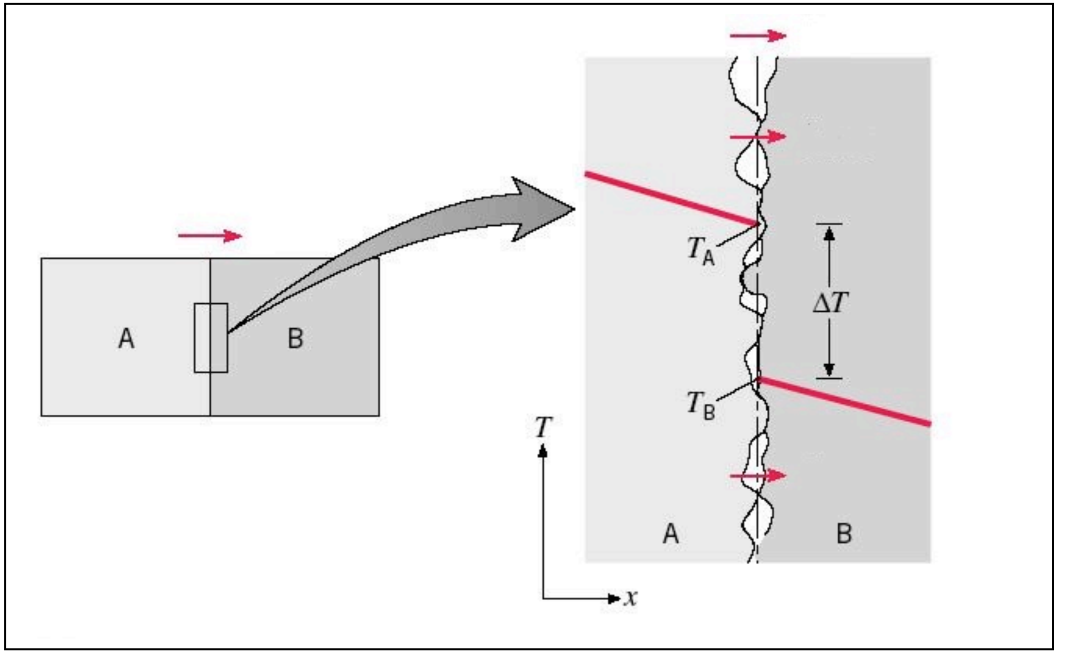
$$q_x \equiv U \cdot A \cdot \Delta T$$

$$U = \frac{1}{\dfrac{1}{h_1} + \dfrac{L_A}{\lambda_A} + \dfrac{L_B}{\lambda_B} + \dfrac{L_C}{\lambda_C} + \dfrac{1}{h_2}}$$

$$q_x = \frac{T_{\infty 1} - T_{\infty 2}}{R_{tot}} = UA(T_{\infty 1} - T_{\infty 2})$$

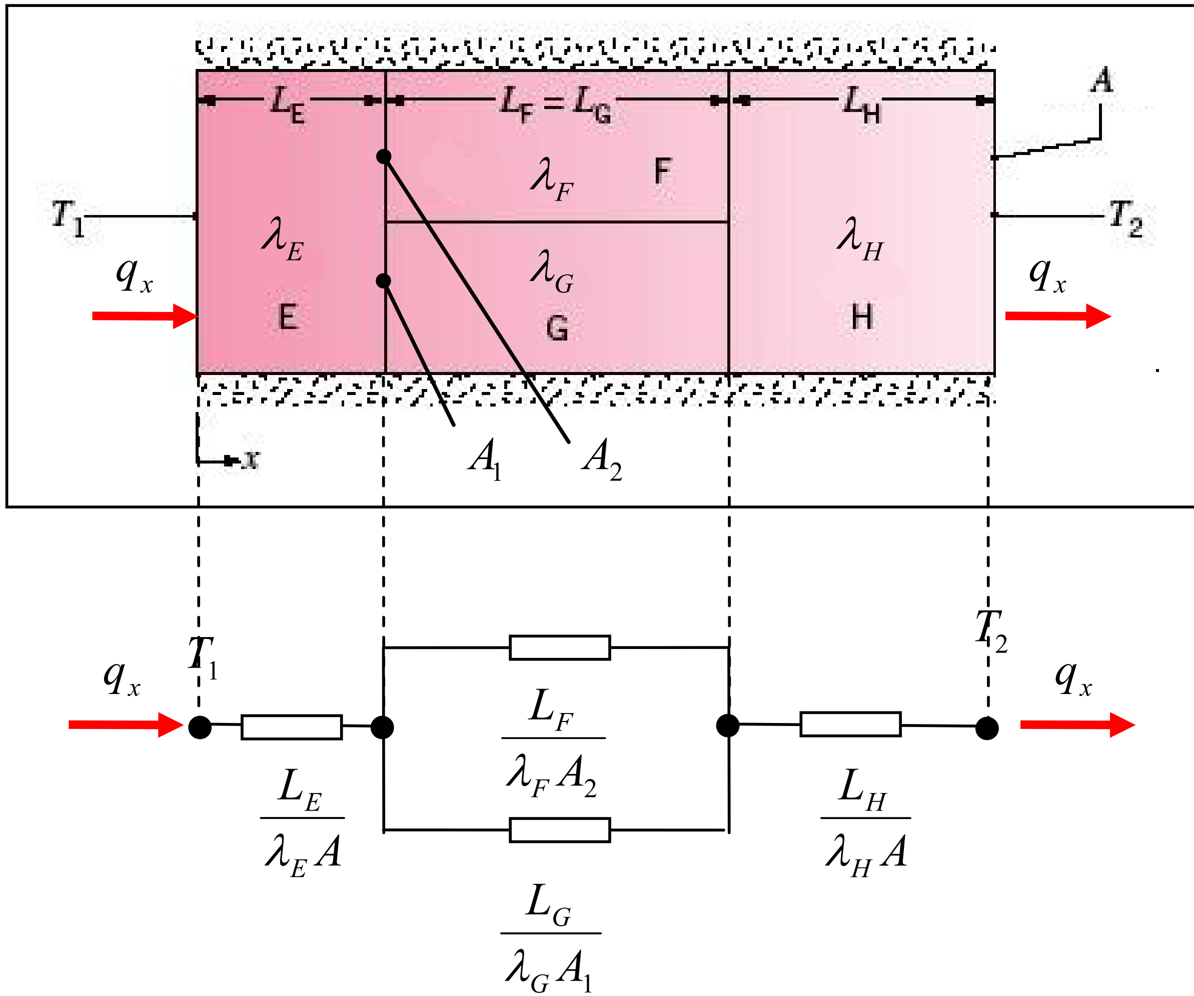
$$\varphi_x = U(T_{\infty 1} - T_{\infty 2}) = \frac{T_{\infty 1} - T_{\infty 2}}{\dfrac{1}{h_1} + \dfrac{L_A}{\lambda_A} + \dfrac{L_B}{\lambda_B} + \dfrac{L_C}{\lambda_C} + \dfrac{1}{h_2}}$$

- **Résistance de contact**

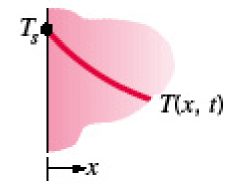
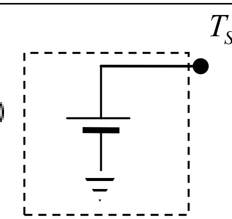
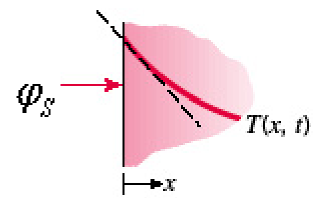
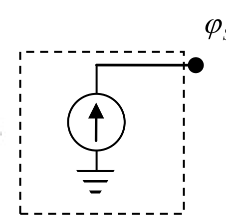
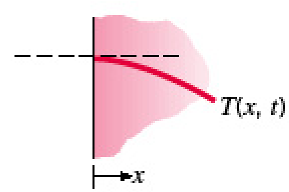
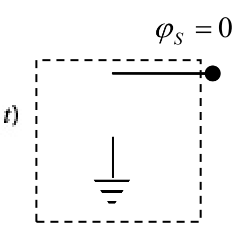
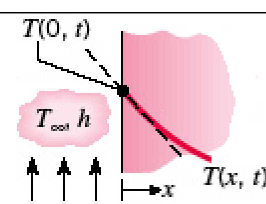
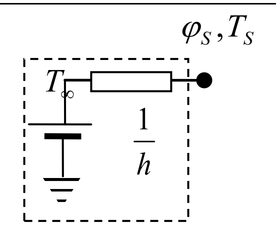


$$R_{ct} \equiv \frac{T_A - T_B}{q_x}$$

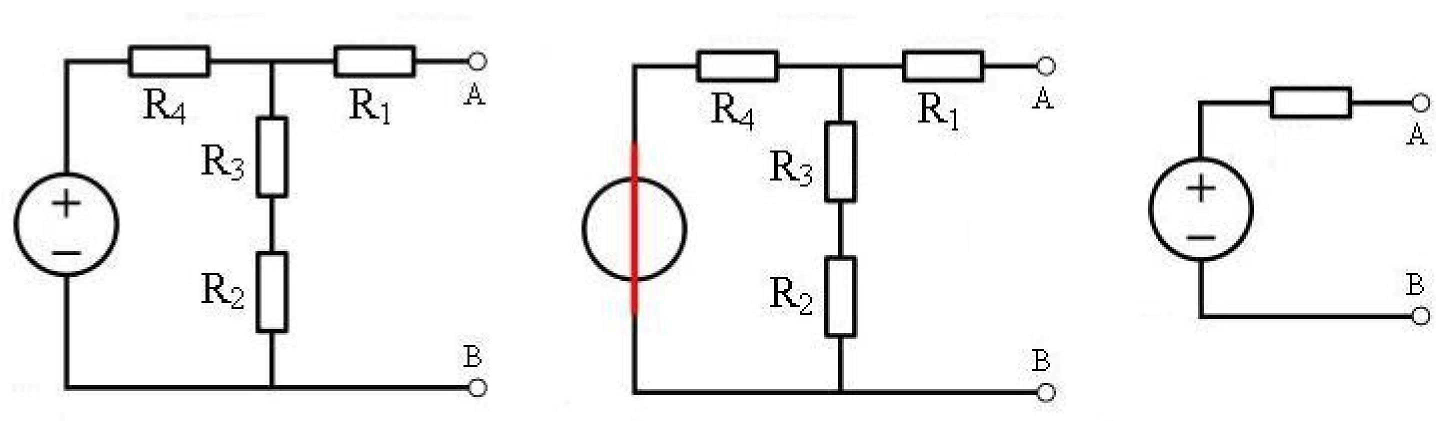
- **Exemple de circuit**



Conduction unidimensionnelle régime stationnaire
Circuits thermiques

	Type de condition	Equation	Variation de la température	Source équivalente
1.	Température imposée sur la surface (condition de Dirichlet)	$T(0,t) = T_S$		
2.	Densité de flux imposée sur la surface (condition de Neumann)			
	a. Densité de flux imposée	$-\lambda \dfrac{\partial T}{\partial x}\Big\vert_{x=0} = \varphi_S$		
	b. Surface adiabatique ou surface de symétrie	$\dfrac{\partial T}{\partial x}\Big\vert_{x=0} = \varphi_S = 0$		
3.	Surface avec convection (condition de Fourier)	$-\lambda \dfrac{\partial T}{\partial x}\Big\vert_{x=0} \equiv \varphi_S$ $= h[T_\infty - T(0,t)]$ $= h(T_\infty - T_S)$		

- **Thévenin**
  - tension ← diff. potentiel aux bornes
  - résistance ← sources passives

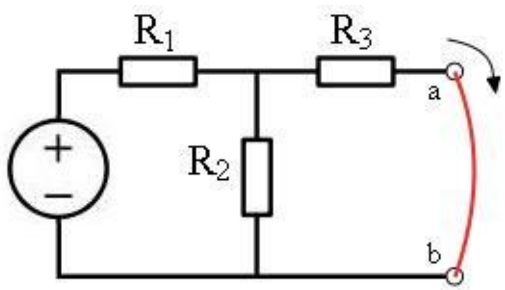


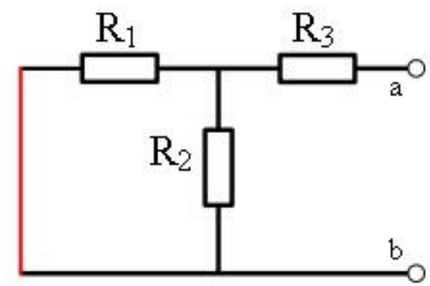
$$T_T = \frac{R_2 + R_3}{(R_2 + R_3) + R_4} T_1$$

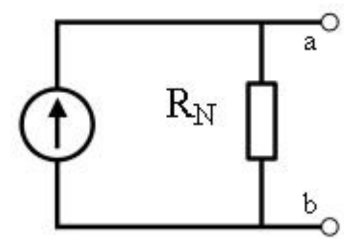
$$q_{tot} = \frac{T}{R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}}$$

$$R_T = R_1 + \left(\frac{1}{R_2 + R_3} + \frac{1}{R_4}\right)^{-1}$$

- **Norton**
  - courant ← bornes en court circuit
  - résistance ← sources passives





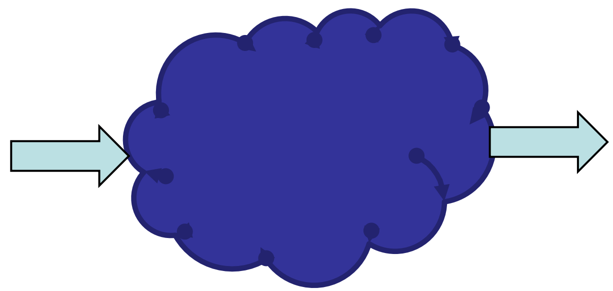


$$q_N = \frac{R_2}{R_2 + R_3} q_{tot}$$

$$R_N = R_3 + \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}$$

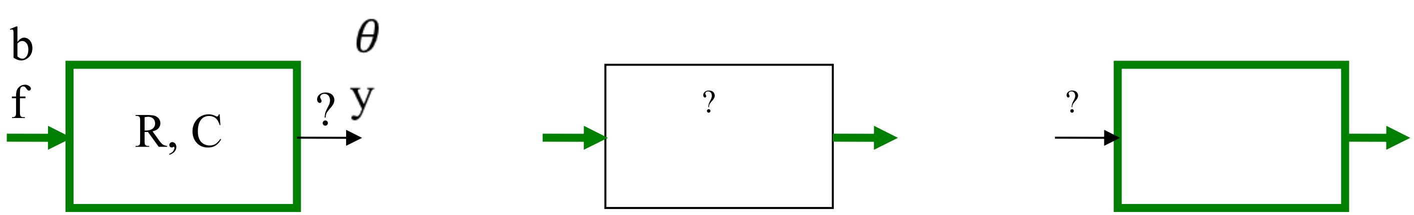
Résolution numériques des problèmes directs

Types de problèmes



- **directe** **inverse**

dimensionnement **contrôle-commande**



(*a*) (*b*) (*c*)

- **Equation de la chaleur**

$$\rho c \frac{\partial T}{\partial t} = \lambda \, div(\mathbf{grad}T) + p$$

- **Matériel homogène isotrope** $\lambda = const.$

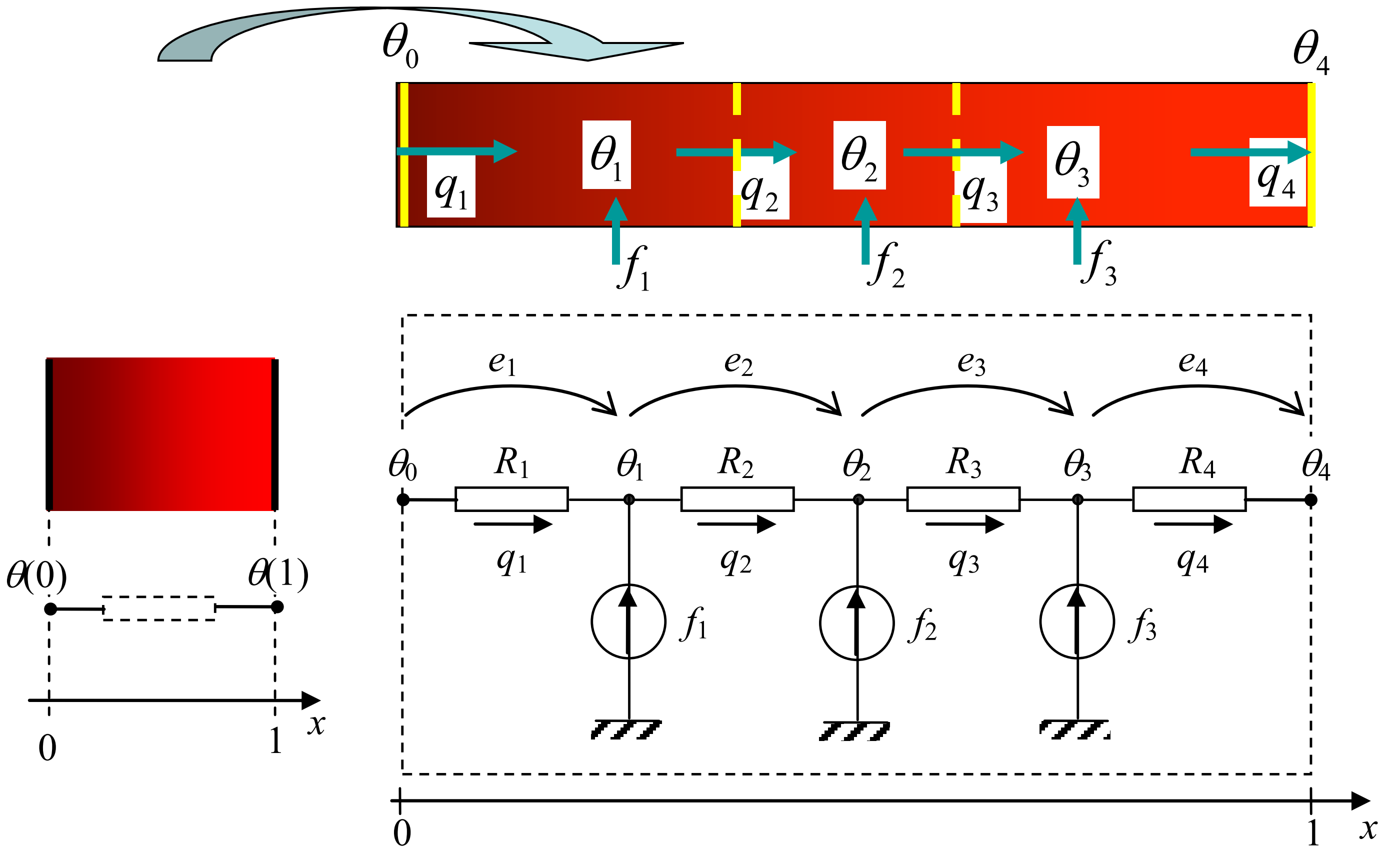
en régime stationnaire $\frac{\partial T}{\partial t} = 0$

équation de Poisson :

$$\lambda \, div(\mathbf{grad}T) + p = 0 \qquad \lambda \, \Delta T + p = 0$$

- **Equation de Poisson**

$$\mathrm{div}(-\lambda\,\mathbf{grad}\,\theta) = p$$



$\mathrm{div}(-\lambda\,\mathbf{grad}\,\theta) = p$

$\mathbf{K}\cdot\boldsymbol{\theta} = \mathbf{f}$

(*a*) (*b*)

Conduction

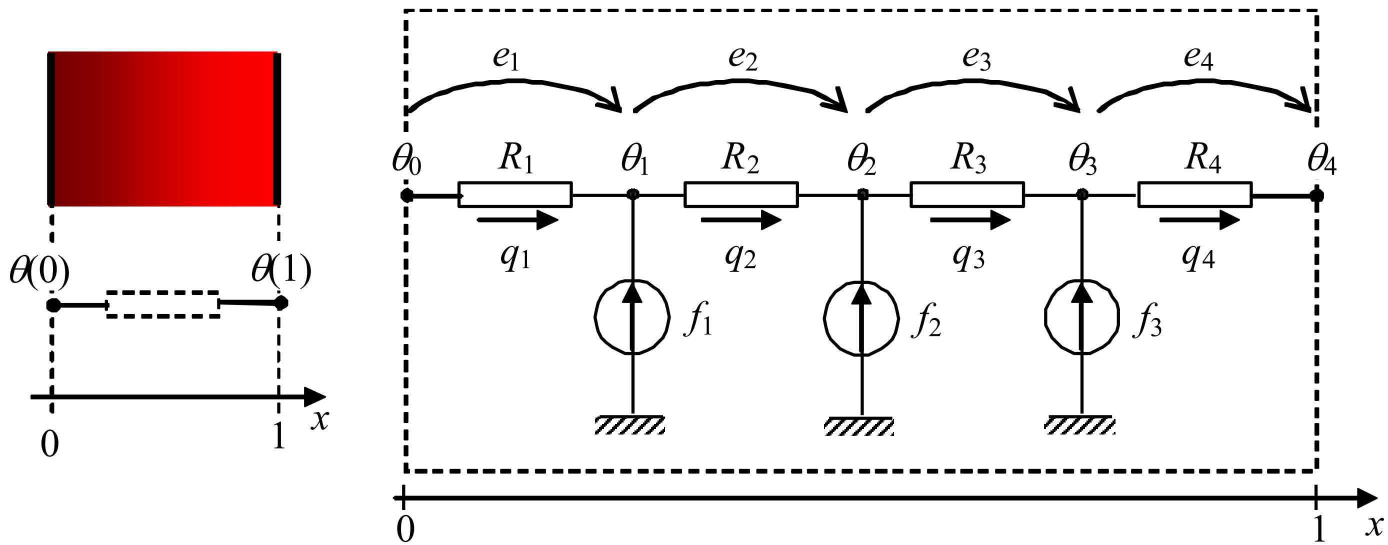
Sol. numérique 1D

- Types problèmes
- Equation chaleur
- Formes particulaires
- Conditions limites

Conduction 1D

- Sans sources
- Résistance therm.
- Avec sources
- Circuits thermiques

- **Problème direct**

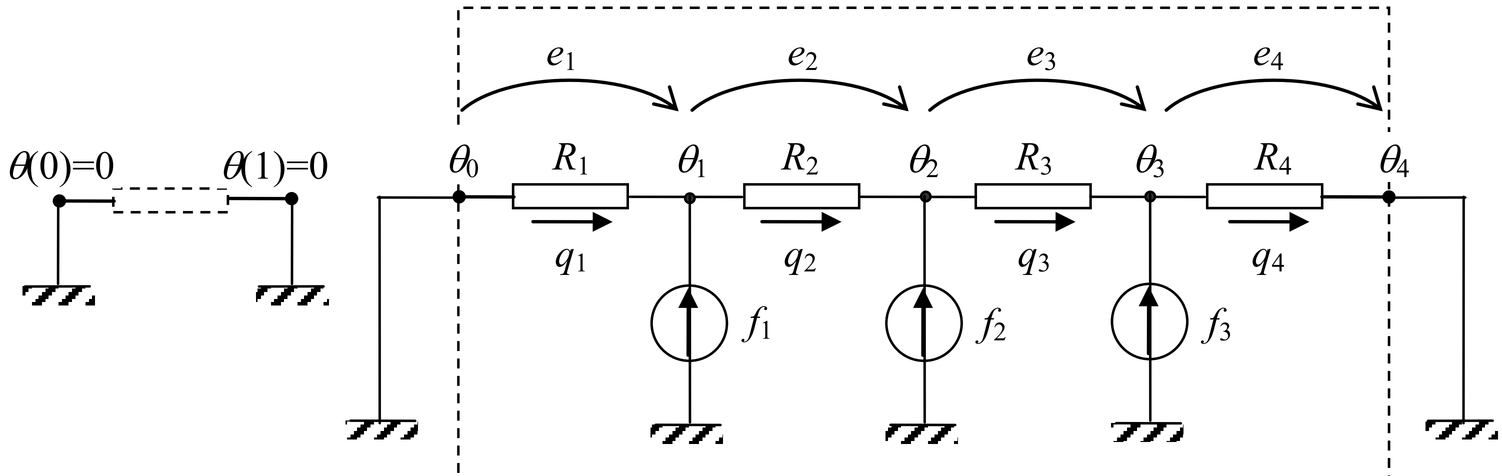


$$\mathrm{div}(\lambda\ \mathbf{grad}\,\theta) = -p \qquad\qquad \mathbf{A}^T\mathbf{C}\mathbf{A}\cdot\boldsymbol{\theta} = -\mathbf{f}$$

(*a*) (*b*)

- données : circuit et conditions aux limites
- trouver : températures des nœuds

- **Conditions de Dirichlet aux deux limites**



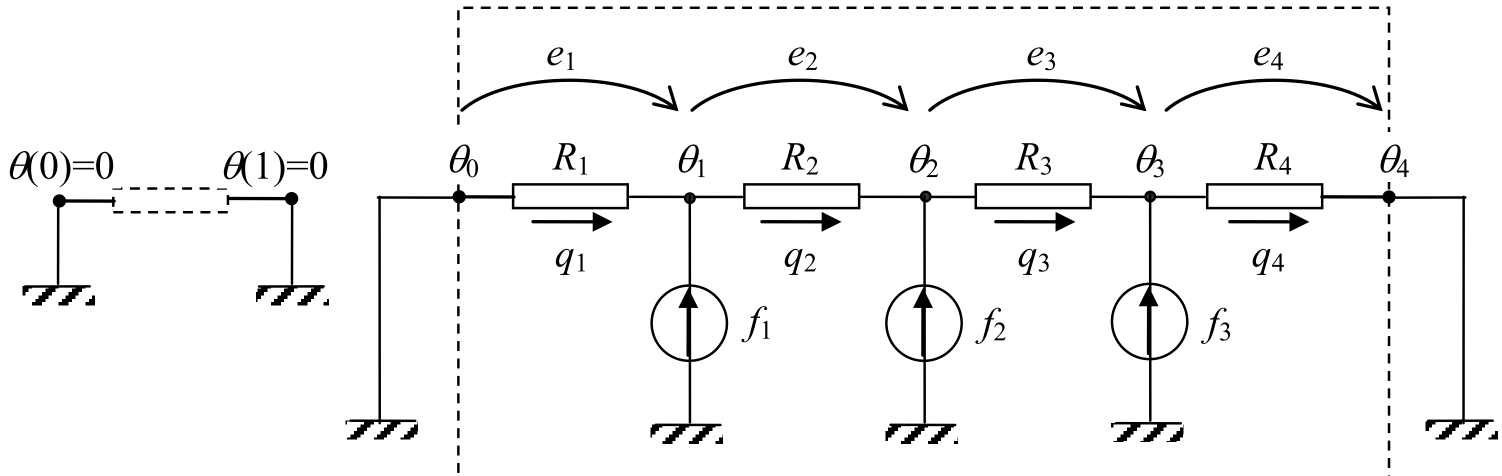
  – différence de température pour chaque résistance

$$\begin{cases} e_1 = -\theta_1 \\ e_2 = \theta_1 - \theta_2 \\ e_3 = \theta_2 - \theta_3 \\ e_4 = \theta_3 \end{cases} \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \qquad \mathbf{e} = -\mathbf{A} \cdot \boldsymbol{\theta}$$

Conduction

- Sol. numérique 1D
  - Loi Fourier
  - Dirichlet
  - Formes particulaires
  - Conditions limites
- Conduction 1D
  - Sans sources
  - Résistance therm.
  - Avec sources
  - Circuits thermiques

• Conditions de Dirichlet aux deux limites

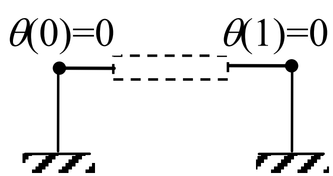


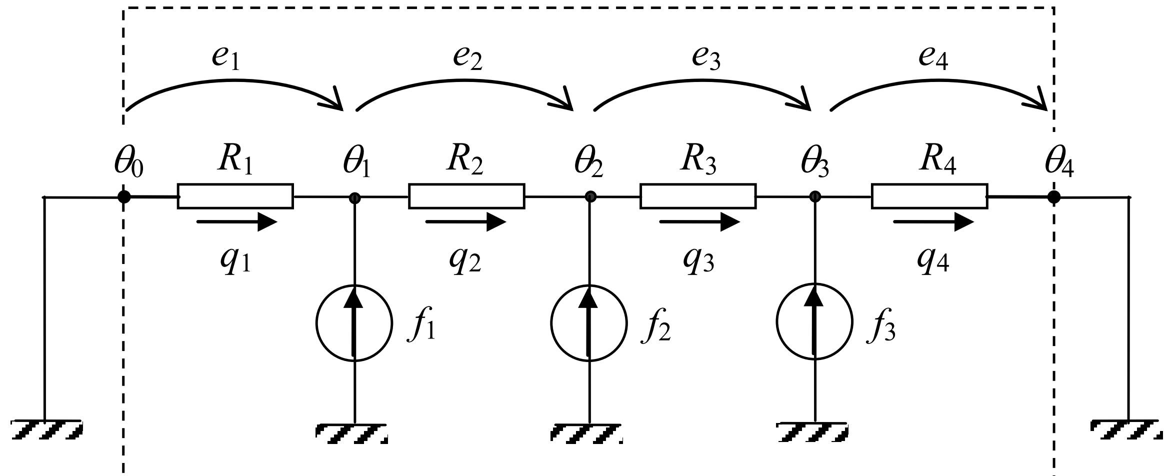
– matrice de connexions : topologie du circuit

$$\begin{array}{c} \text{nœud} \\ \\ \mathbf{A}_0 = \end{array} \begin{array}{c} \begin{array}{ccccc} \theta_0 & \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{arc} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}$$

$$\begin{array}{c} \text{nœud} \\ \\ \mathbf{A} = \end{array} \begin{array}{c} \begin{array}{ccc} \theta_1 & \theta_2 & \theta_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \end{array} \begin{array}{c} \text{arc} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}$$

- **Conditions de Dirichlet aux deux limites**



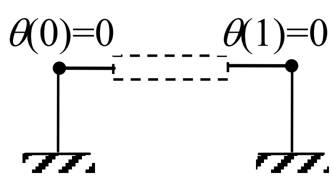


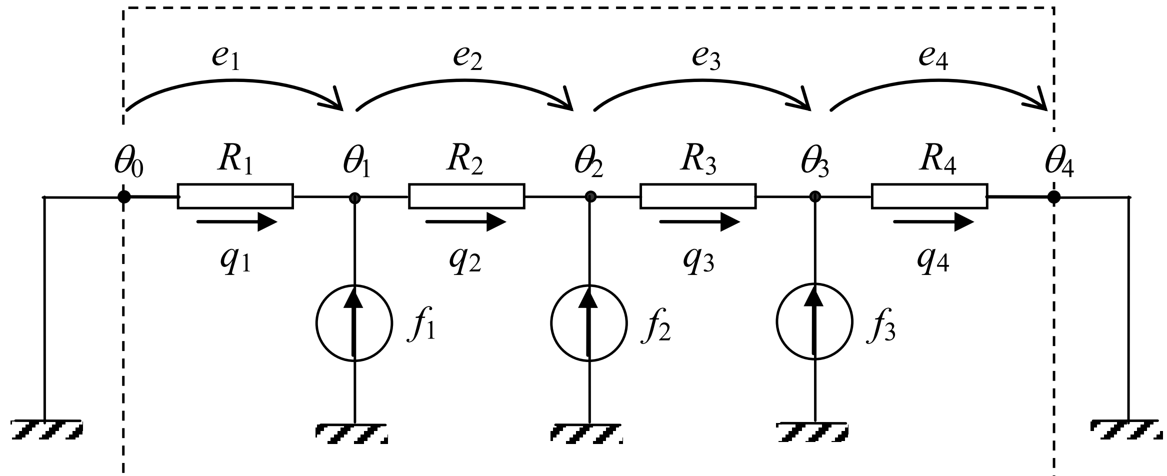
  - flux : loi constitutive (Fourier)

$$\begin{cases} q_1 = G_1 e_1 \\ q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \\ q_4 = G_4 e_4 \end{cases} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & G_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

$$G_i = 1/R_i,\ i = 1,\ldots,4$$

- **Conditions de Dirichlet aux deux limites**





– bilan de flux : loi Kirchhoff

$$\begin{cases} q_1 - q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 - q_4 = -f_3 \end{cases} \qquad -\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \qquad -\mathbf{f} = \mathbf{A}^T \cdot \mathbf{q}$$

Résolution numériques des problèmes directs
Problèmes directs en conduction stationnaire

Conduction

Sol. numérique 1D

Loi Fourier

Dirichlet

Formes particulaires

Conditions limites

Conduction 1D

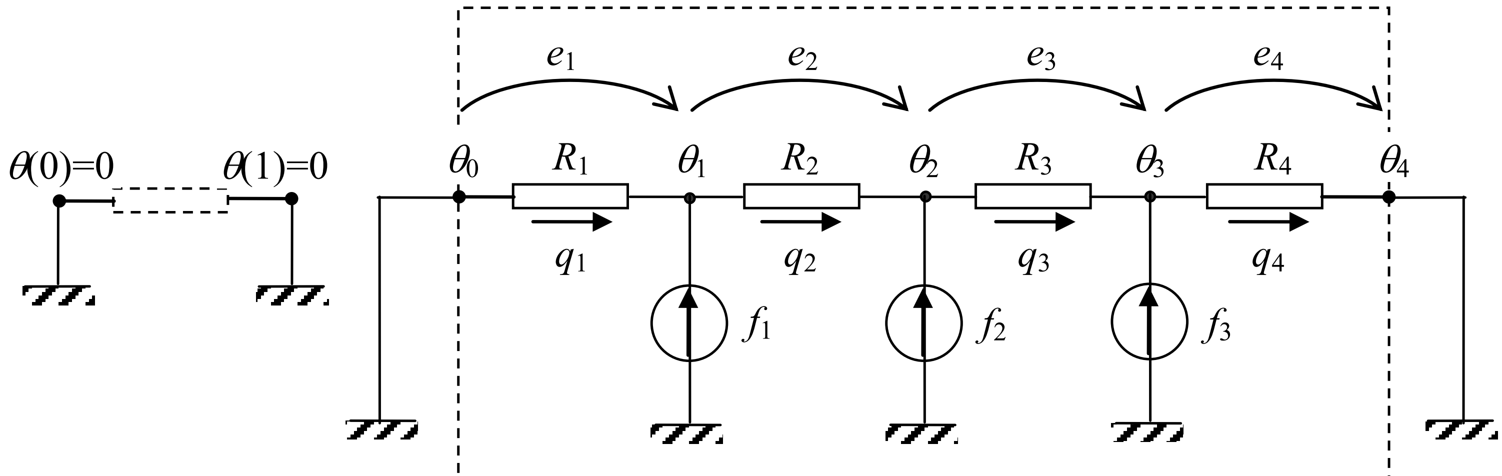
Sans sources

Résistance therm.

Avec sources

Circuits thermiques

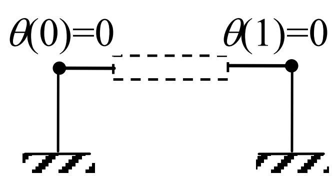
- **Conditions de Dirichlet aux deux limites**

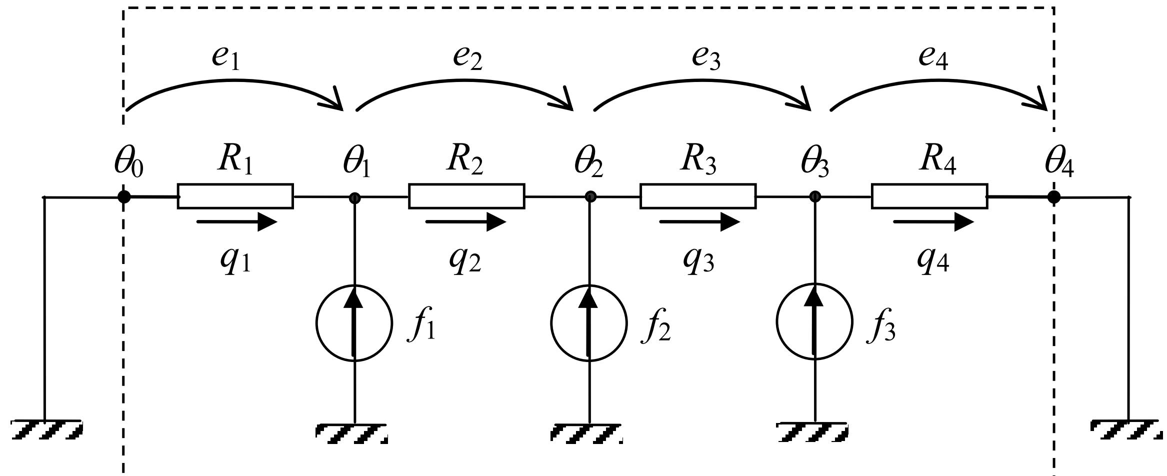


  – équation du circuit

$$\mathbf{K} \cdot \boldsymbol{\theta} = \mathbf{f} \qquad \mathbf{K} = \mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A} = \begin{bmatrix} G_1 + G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 & -G_3 \\ 0 & -G_3 & G_3 + G_4 \end{bmatrix}$$

- **Conditions de Dirichlet aux deux limites**





$$\mathbf{f} = \mathbf{K}\cdot\boldsymbol{\theta} \longrightarrow \boldsymbol{\theta} = \mathbf{K}^{-1}\cdot\mathbf{f}$$

$$\mathbf{G} = \mathbf{I}$$

$$\mathbf{K} \equiv \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Résolution numériques des problèmes directs
Problèmes directs en conduction stationnaire

- **Conditions de Dirichlet aux deux limites**

$$\mathbf{K} \equiv \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- **Propriétés de la matrice K**
  - symétrique $\mathbf{K} = \mathbf{K}^T$
  - tri-diagonale
  - creuse
  - diagonale constante

- **Matrice de Toeplitz**

- **Analyse nodale des circuits**

$$
\begin{tikzcd}
\boldsymbol{\theta} \arrow[d, "-\mathbf{A}", "\text{Connexions}"'] & & -\mathbf{A}^T\mathbf{q} = \mathbf{f} \\
\mathbf{e} = -\mathbf{A}\boldsymbol{\theta} \arrow[rr, "\text{Loi constitutive}", "\mathbf{G}"'] & & \mathbf{q} = \mathbf{G}\mathbf{e} \arrow[u, "\text{Bilans}", "-\mathbf{A}^T"']
\end{tikzcd}
$$

$\boldsymbol{\theta}$	températures dans les nœuds
$\mathbf{e}$	chute de température sur les résistances
$\mathbf{q}$	flux dans les résistances
$\mathbf{f}$	flux externes

Conduction

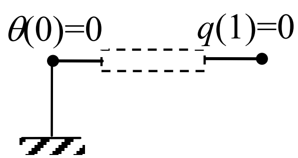
- Sol. numérique 1D
  - Loi Fourier
  - Dirichlet
  - Formes particulaires
  - Conditions limites
- Conduction 1D
  - Sans sources
  - Résistance therm.
  - Avec sources
  - Circuits thermiques

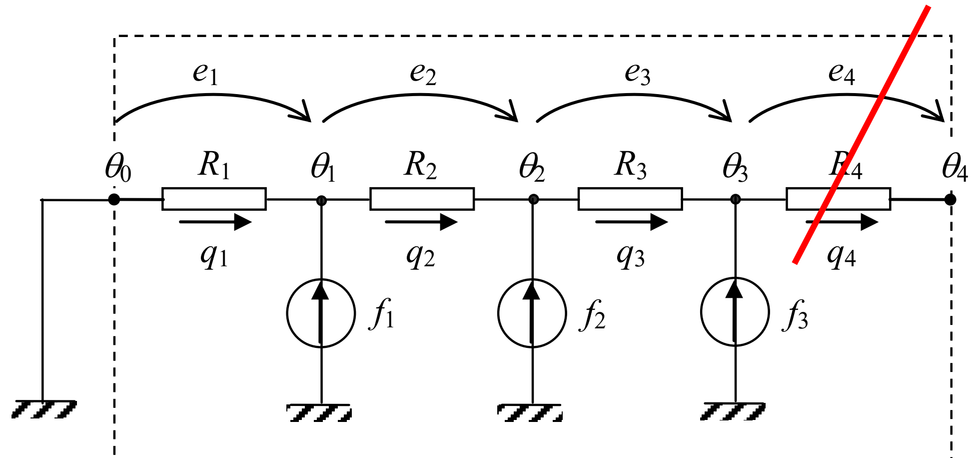
- **Equivalence entre le modèle continu et discret**

$$
\begin{tikzcd}
T \arrow[d, "\mathbf{grad}", "\text{Connexion}"'] & & f = p - \rho c \dfrac{\partial T}{\partial t} \\
\nabla T \arrow[rr, "\text{Loi constitutive}", "-\lambda"'] & & \boldsymbol{\varphi} \arrow[u, "\text{Bilan}", "div"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{\theta} \arrow[d, "\text{Connexions}"', "-\mathbf{A}"] & & \mathbf{A}^T\mathbf{q} = \mathbf{f} \\
\mathbf{e} = -\mathbf{A}\boldsymbol{\theta} \arrow[rr, "\text{Loi constitutive}", "\mathbf{G}"'] & & \mathbf{q} = \mathbf{G}\mathbf{e} \arrow[u, "\text{Bilans}", "-\mathbf{A}^T"']
\end{tikzcd}
$$

	Continu	Discret
Equation de la chaleur	$\text{div}(\lambda\, \mathbf{grad}\,\theta) = -p$	$\mathbf{A}^T\mathbf{G}\mathbf{A}\,\boldsymbol{\theta} = \mathbf{f}$
Conditions aux limites	en $x = 0 : \theta(0)$ ou $\varphi(0)$ en $x = 1 : \theta(1)$ ou $\varphi(1)$	incluses dans la matrice $\mathbf{A}^T\mathbf{G}\mathbf{A}$
Température	θ	$\boldsymbol{\theta}$
Flux	φ	$\mathbf{q}$
Connexion	$-\mathbf{grad}\,\theta$	$-\mathbf{A}\boldsymbol{\theta}$
Bilan d'énergie	-div φ	$-\mathbf{A}^T\mathbf{q}$
Constante de matériau	λ	$\mathbf{G}$
Sources	p	$-\mathbf{f}$

- **Dirichlet plus Neumann**





  – différence de température pour chaque résistance

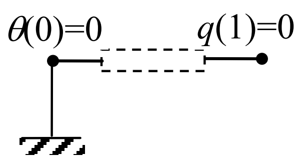
$$\begin{cases} e_1 = -\theta_1 \\ e_2 = \theta_1 - \theta_2 \\ e_3 = \theta_2 - \theta_3 \end{cases} \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \qquad \mathbf{e} = -\mathbf{A} \cdot \boldsymbol{\theta}$$

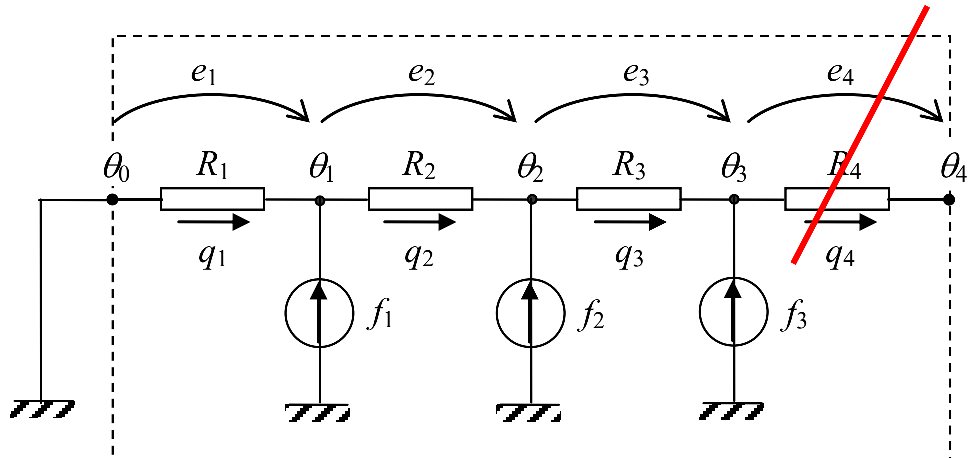
$$e_4 = 0 \qquad \text{(connue)}$$

Résolution numériques des problèmes directs

Problèmes directs en conduction stationnaire

- **Dirichlet plus Neumann**



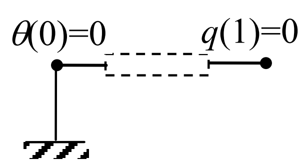


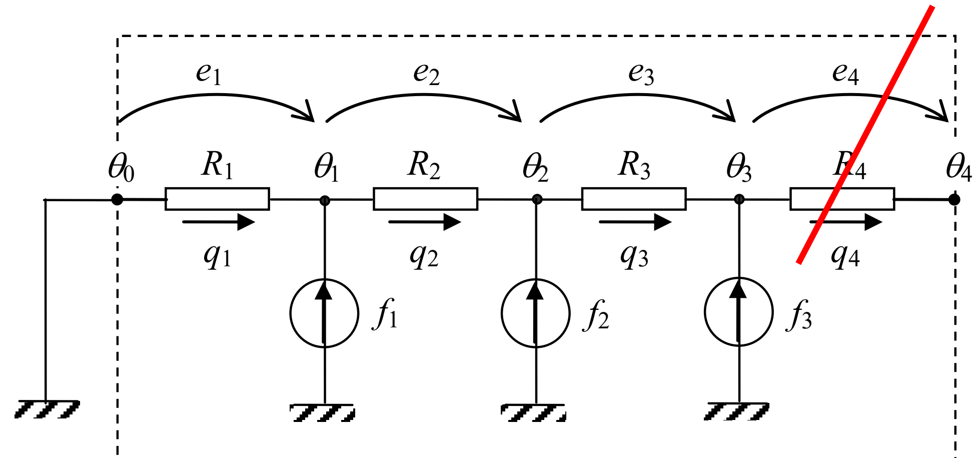
  – matrice de connexions : topologie du circuit

$$
\begin{array}{lccc}
\text{nœud} & \theta_1 & \theta_2 & \theta_3
\end{array}
$$

$$
\mathbf{A} = \begin{array}{c}\left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] \end{array}
\begin{array}{l} \text{arc} \\ q_1 \\ q_2 \\ q_3 \\ \cancel{q_4} \end{array}
$$

- **Dirichlet plus Neumann**





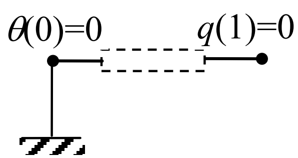
– flux : loi constitutive

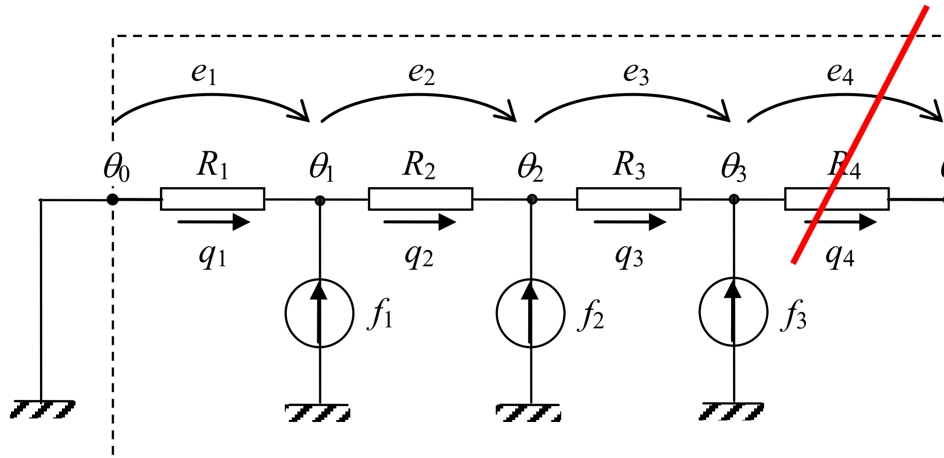
$$\begin{cases} q_1 = G_1 e_1 \\ q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \end{cases} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

$$q_4 = G_4 e_4 = 0 \qquad \text{(connu)}$$

$$G_i = 1 / R_i,\ i = 1,\ldots,4$$

- **Dirichlet plus Neumann**





  – bilan de flux : loi Kirchhoff

$$\begin{cases} q_1 - q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 - q_4 = -f_3 \end{cases} \qquad -\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad -\mathbf{f} = \mathbf{A}^T \cdot \mathbf{q}$$

Conduction

Sol. numérique 1D

Loi Fourier

Equation chaleur

Dirichlet & Neumann

Conditions limites

Conduction 1D

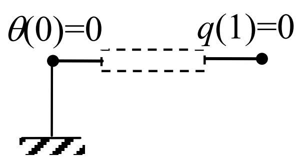
Sans sources

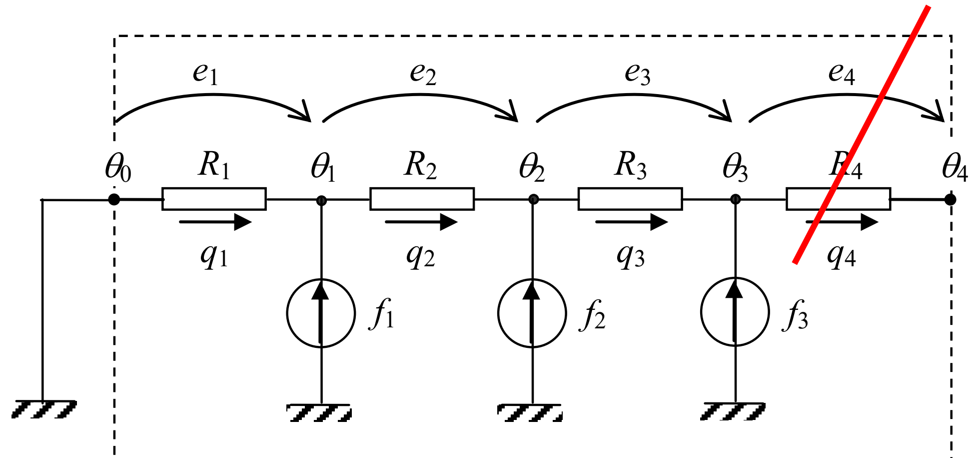
Résistance therm.

Avec sources

Circuits thermiques

- **Dirichlet plus Neumann**



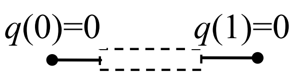


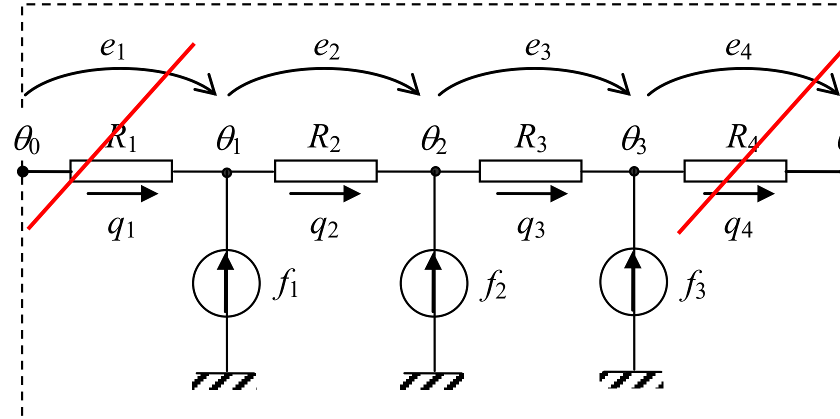
– équation du circuit

$$\mathbf{f} = \mathbf{H} \cdot \boldsymbol{\theta} \qquad \mathbf{G} = \mathbf{I} \qquad \mathbf{H} \equiv \mathbf{A}^T \mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{\theta} = \mathbf{K}^{-1} \cdot \mathbf{f}$$

- **Conditions de Neumann aux deux limites**





  – différence de température pour chaque résistance

$$\begin{cases} e_1 = 0 \\ e_2 = \theta_1 - \theta_2 \\ e_3 = \theta_2 - \theta_3 \\ e_4 = 0 \end{cases}$$

$$\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

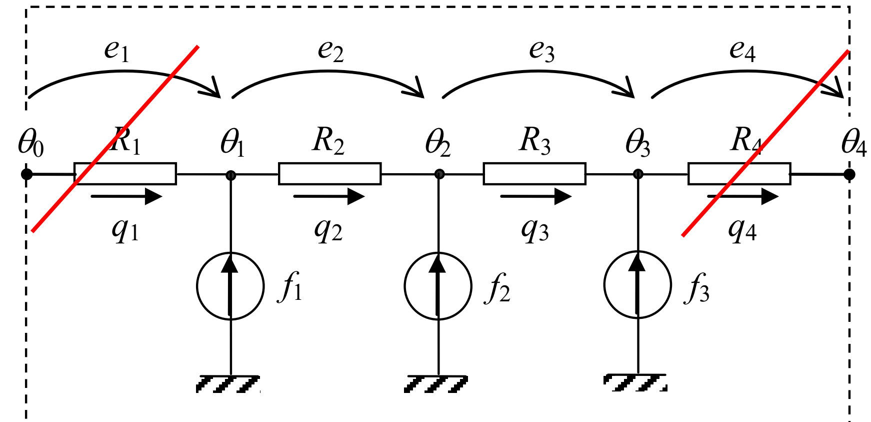
$$\mathbf{e} = -\mathbf{A} \cdot \boldsymbol{\theta}$$

Résolution numériques des problèmes directs
Problèmes directs en conduction stationnaire

Conduction

- **Sol. numérique 1D**
  - Loi Fourier
  - Equation chaleur
  - Formes particulaires
  - **Neumann**
- Conduction 1D
  - Sans sources
  - Résistance therm.
  - Avec sources
  - Circuits thermiques

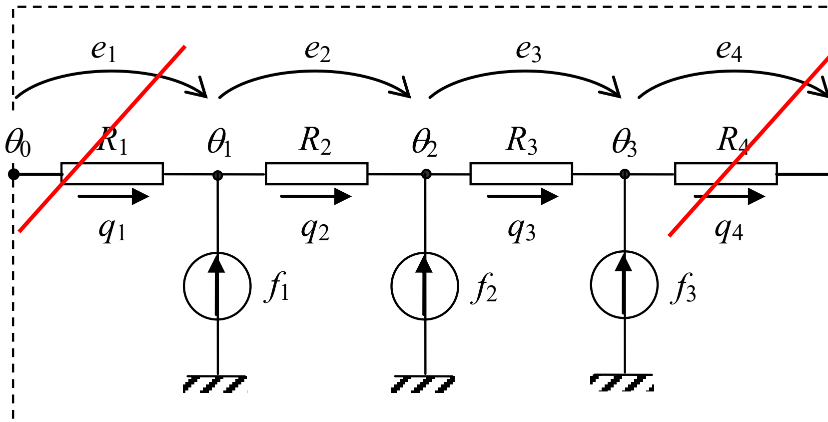
- **Conditions de Neumann aux deux limites**



– matrice de connexions

$$
\begin{array}{cccc|c}
\text{nœud} & \theta_1 & \theta_2 & \theta_3 & \text{arc} \\
 & & & & \cancel{R_1} \\
\mathbf{A} = & \left[\begin{array}{c} -1 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ -1 \end{array} & \left.\begin{array}{c} 0 \\ 1 \end{array}\right] & \begin{array}{c} R_2 \\ R_3 \end{array} \\
 & & & & \cancel{R_4}
\end{array}
$$

- **Conditions de Neumann aux deux limites**



- flux : loi constitutive

$$q_1 = 0$$

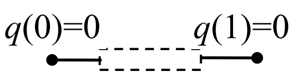
$$\begin{cases} q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \end{cases} \qquad \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} G_2 & 0 \\ 0 & G_3 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

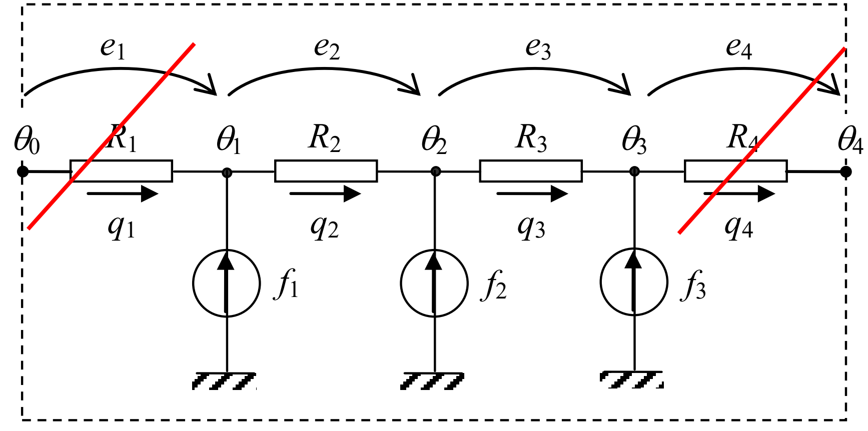
$$q_4 = 0$$

$$G_i = 1 / R_i, \, i = 1, \ldots, 4$$

Résolution numériques des problèmes directs
Problèmes directs en conduction stationnaire

- **Conditions de Neumann aux deux limites**



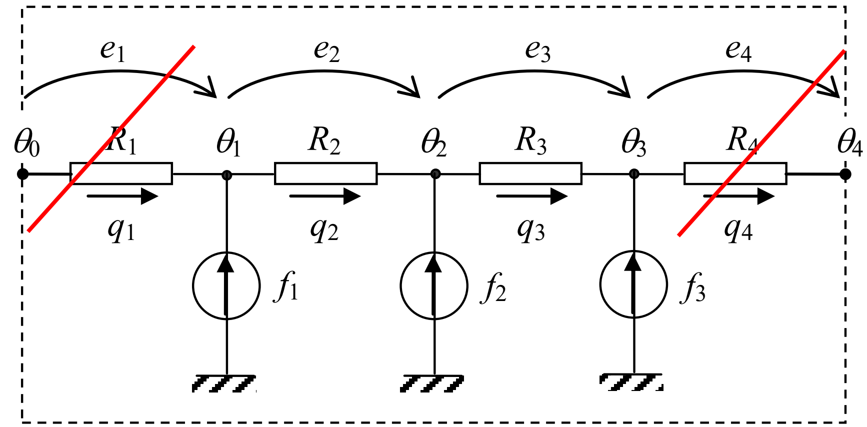


  – bilan de flux : loi de Kirchhoff

$$\begin{cases} -q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 = -f_3 \end{cases} \qquad -\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} \qquad -\mathbf{f} = \mathbf{A}^T \cdot \mathbf{q}$$

Résolution numériques des problèmes directs
Problèmes directs en conduction stationnaire

- **Conditions de Neumann aux deux limites**



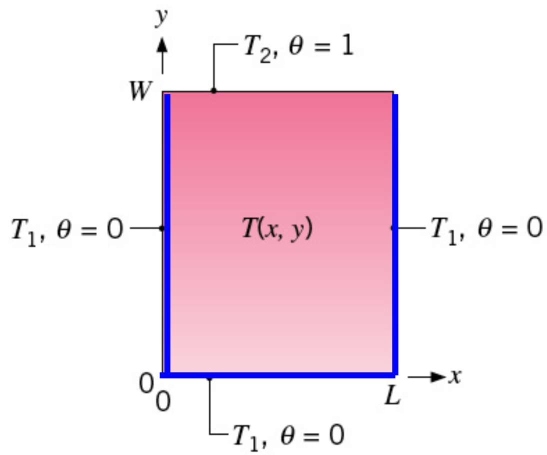
– équation du circuit

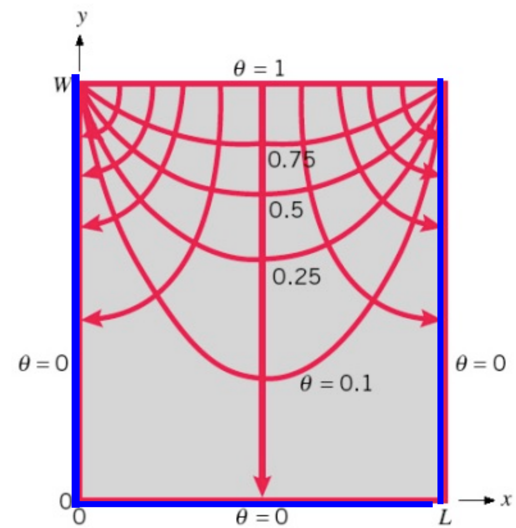
$$\mathbf{f} = \mathbf{B}\cdot\boldsymbol{\theta} \qquad \mathbf{C} = \mathbf{I} \qquad \mathbf{B} \equiv \mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

singulière ou *non inversable*

$\boxed{\text{Résolution : nécessite température de référence}}$

- **2D, stationnaire, sans sources internes**





$$\rho c \frac{\partial T}{\partial t} = div(\lambda \, \mathbf{grad} T) + p \longrightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Eq. Laplace

Conduction bidimensionnelle régime stationnaire

Solution analytique : séparation des variables

Conduction

Conduction - général

Loi Fourier

Equation chaleur

Formes particulaires

Conditions limites

Conduction 2D

Analytique

Résistance therm.

Avec sources

Circuits thermiques

- **2D, stationnaire, sans sources internes**

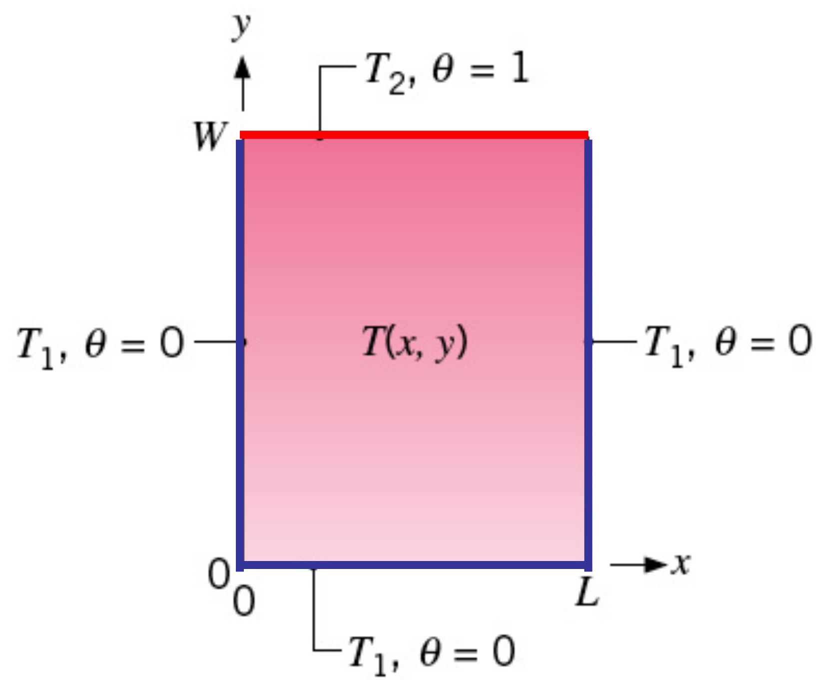
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(0, y) = T_1$$

$$T(L, y) = T_1$$

$$T(x, 0) = T_1$$

$$T(x, W) = T_2$$



$$\theta = \frac{T - T_1}{T_2 - T_1}$$

- **Changement de variable**

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(0,y) = T_1$$

$$T(L,y) = T_1$$

$$T(x,0) = T_1$$

$$T(x,W) = T_2$$

$$\xrightarrow{\theta = \frac{T - T_1}{T_2 - T_1}}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$$\theta(0,y) = 0$$

$$\theta(L,y) = 0$$

$$\theta(x,0) = 0$$

$$\theta(x,W) = 1$$

- **Séparation des variables**
  - supposons la solution de la forme

$$\theta(x,y) = X(x)\cdot Y(y)$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \longrightarrow -\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad X = C_1\cos\lambda x + C_2\sin\lambda x$$

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \qquad Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

$$\boxed{\theta = X\cdot Y = (C_1\cos\lambda x + C_2\sin\lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})}$$

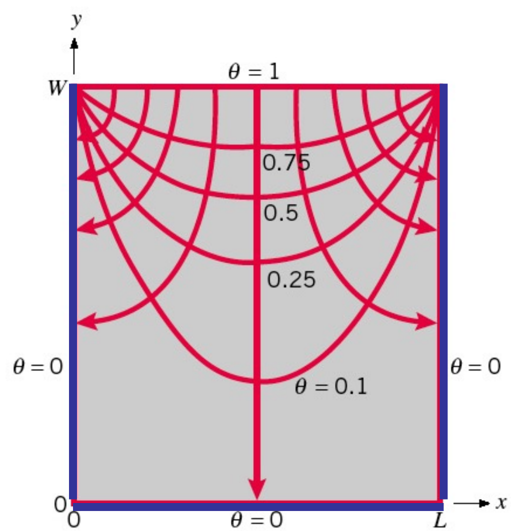
Conduction bidimensionnelle régime stationnaire
Solution analytique : séparation des variables

– solution général

$$\theta = X \cdot Y = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

– conditions aux limites

$$\theta(0, y) = 0 \qquad \theta(L, y) = 0 \qquad \theta(x,0) = 0 \qquad \theta(x, W) = 1$$

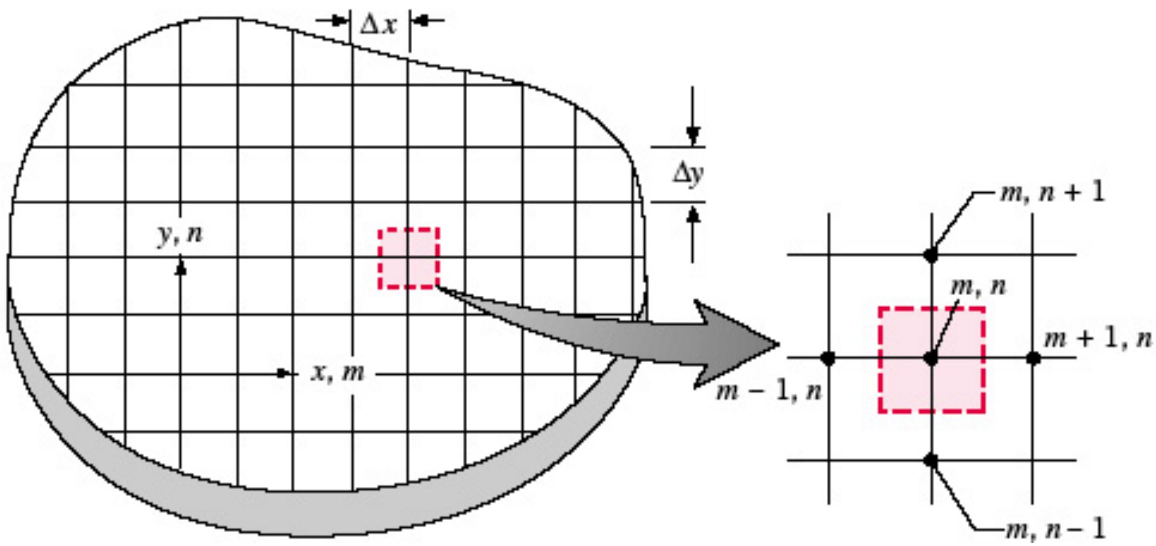


$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n \pi x}{L} \frac{\sinh(n\pi y / L)}{\sinh(n \pi W / L)}$$

Conduction bidimensionnelle régime stationnaire
Solution numérique

- **Différences finies**
  - discrétisation des opération de dérivation
- **Volumes fini**
  - calcul des valeurs moyennes des grandeurs conservatives dans un volume, pas aux nœuds ou surfaces (approximations intégrales)
  - conditions aux limites non-invasives
  - pas besoin de maillage structuré
- **Eléments fini**
  - structures génie civil
  - une fonction définie sur un domaine
  - pas besoin de maillage structuré

- **Différences finies**
  - discrétisation du domaine



  - bilan pour chaque volume de contrôle

$$\dot{E}_e - \dot{E}_s + \dot{E}_g = 0$$

Conduction bidimensionnelle régime stationnaire
Solution numérique

Conduction

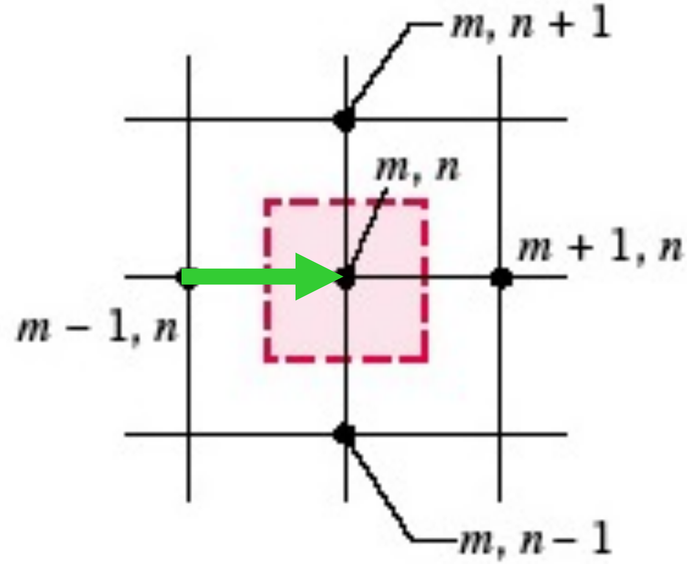
- Conduction - général
  - Loi Fourier
  - Equation chaleur
  - Formes particulaires
  - Conditions limites
- Conduction 2D
  - Sans sources
  - Différences finies
  - Avec sources
  - Circuits thermiques

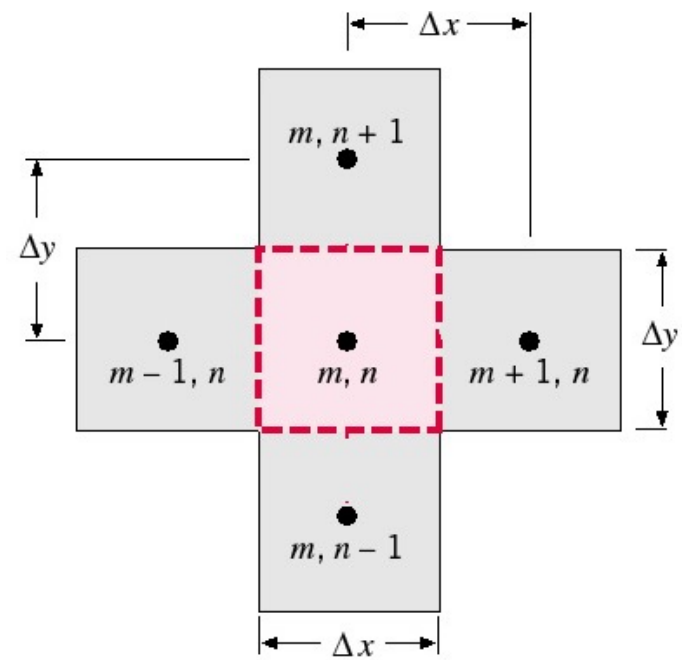
– flux

$$q_x = -\lambda A \frac{dT}{dx}$$

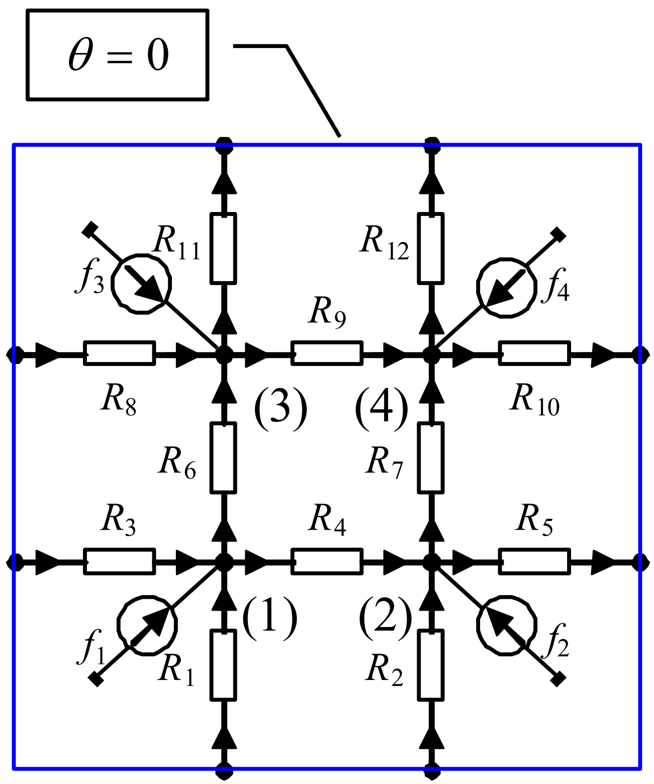
$$q_{(m-1,n)\to(m,n)} = \lambda \underbrace{(\Delta y \cdot 1)}_{A} \underbrace{\frac{\theta_{m-1,n} - \theta_{m,n}}{\Delta x}}_{-\,\mathbf{grad}\,\theta}$$

$$R_{i,j} = \frac{1}{\lambda}\frac{L}{A} = \frac{1}{\lambda}\frac{\Delta x}{\Delta y \cdot 1}$$





- **Conditions Dirichlet**



$$
\begin{array}{c}
\text{nœud} \quad \theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \text{arc} \\
\mathbf{A} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{array}{l}
R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ R_{10} \\ R_{11} \\ R_{12}
\end{array}
\end{array}
$$

- **Conditions Dirichlet**
  - équation du circuit

$$\mathbf{f} = \mathbf{K} \cdot \boldsymbol{\theta}$$

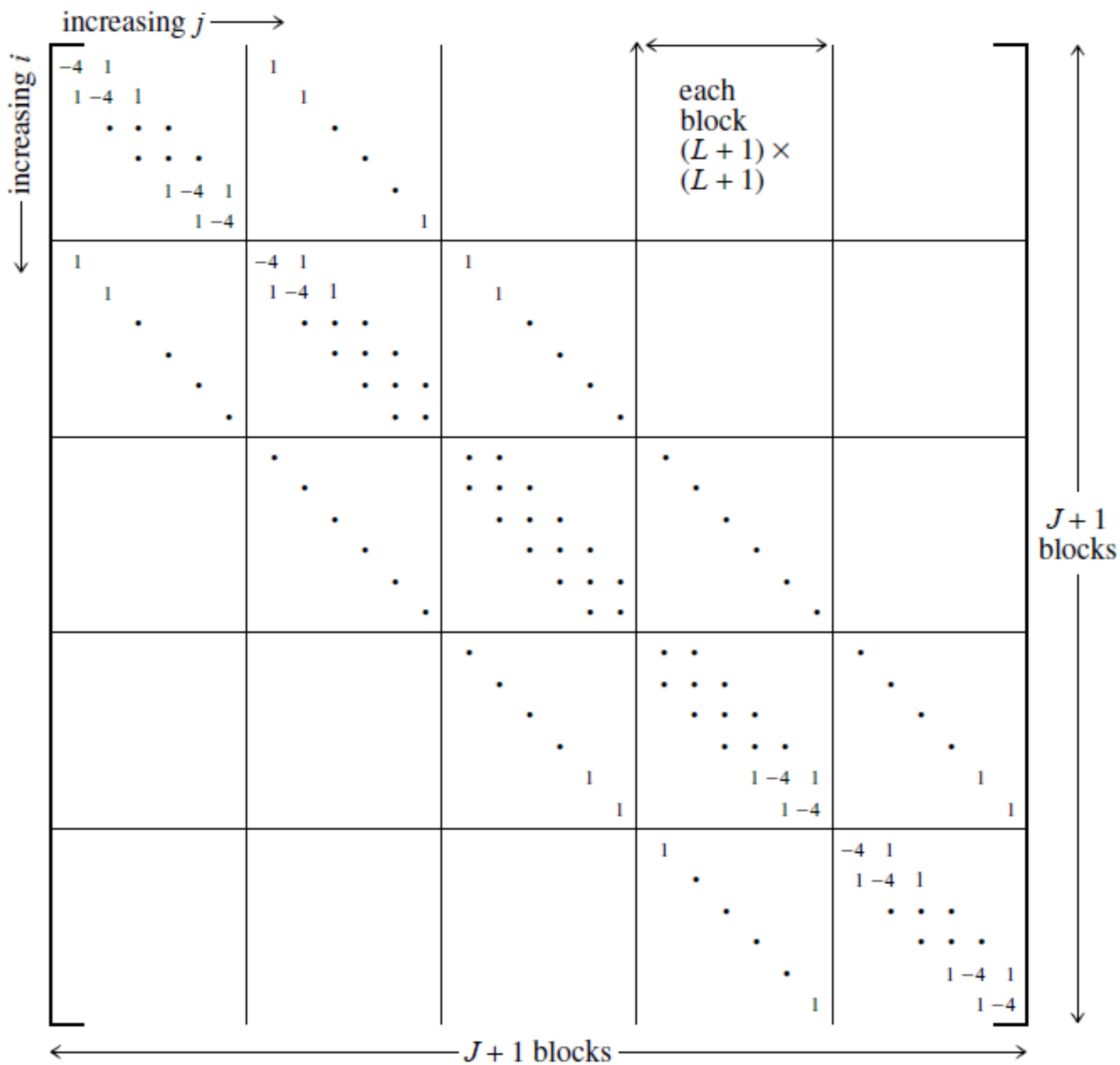
$$f_i = p(\Delta x \cdot \Delta y \cdot 1) \qquad p\ [\mathrm{W/m^3}]$$

$$\mathbf{K} = \mathbf{A}^T \mathbf{G} \mathbf{A}$$

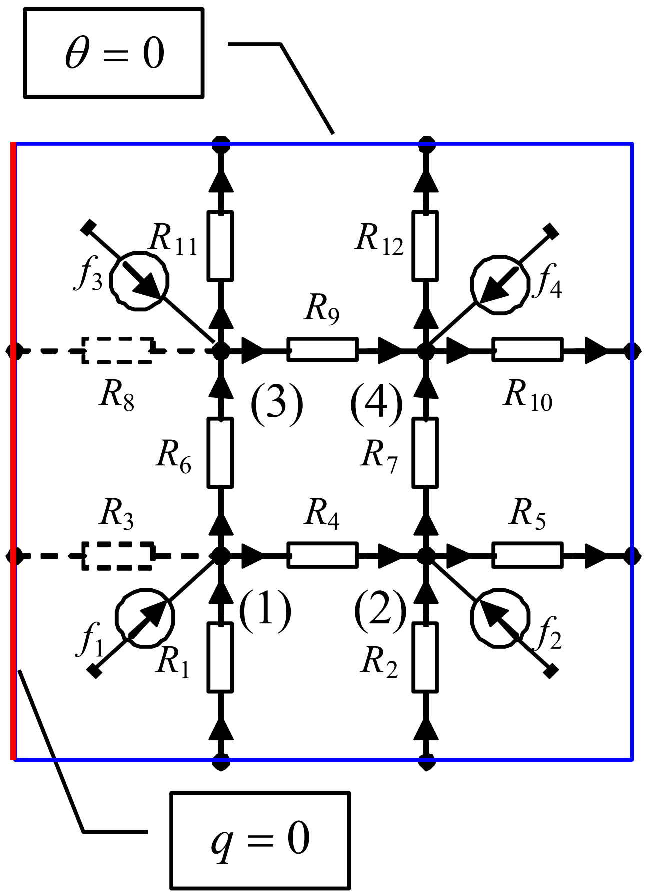
$$\mathbf{K} = \mathbf{A}^T \mathbf{A} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

Conduction bidimensionnelle régime stationnaire

Solution numérique



- **Conditions Dirichlet et Neumann**



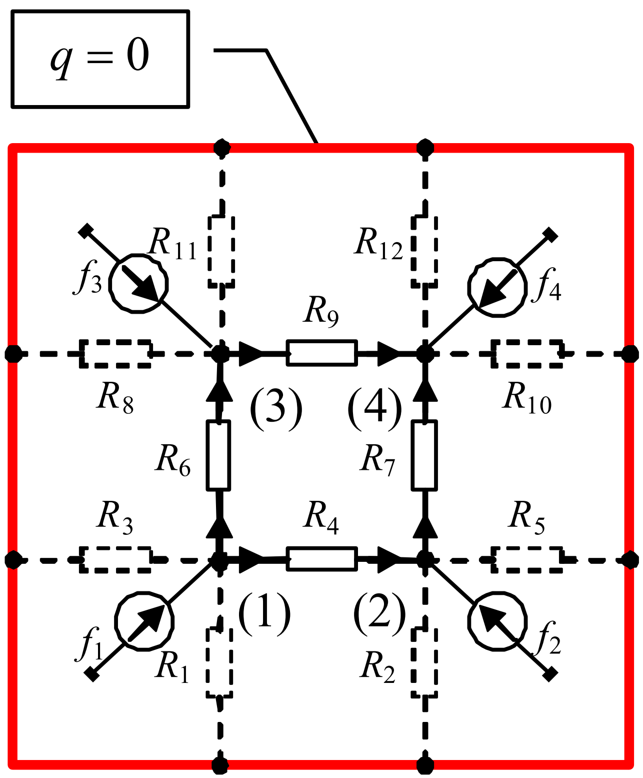
$$\mathbf{A} = \begin{array}{c} \text{nœud} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ \end{array} \text{arc}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \boxed{0} & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & \boxed{0} & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{array}$$

- **Conditions Dirichlet et Neumann**
  - équation du circuit

$$\mathbf{K} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

- **Conditions Neumann**



$$
\mathbf{A} = \begin{array}{c} \text{nœud} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccc} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{array} \\
\begin{bmatrix}
\boxed{0} & 0 & 0 & 0 \\
0 & \boxed{0} & 0 & 0 \\
\boxed{0} & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & \boxed{0} & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & \boxed{0} & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & \boxed{0} \\
0 & 0 & \boxed{0} & 0 \\
0 & 0 & 0 & \boxed{0}
\end{bmatrix}
\end{array}
\begin{array}{c} \text{arc} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{array}
$$

- **Conditions Neumann**

$$\mathbf{K} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

non-inversable

Conduction bidimensionnelle régime stationnaire

Solution numérique

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Dirichlet — Dirichlet & Neumann — Neumann

$$\left.\frac{\partial^2\theta}{\partial x^2}\right|_{m,n} = \frac{\left.\dfrac{\partial\theta}{\partial x}\right|_{m+1/2,n} - \left.\dfrac{\partial\theta}{\partial x}\right|_{m-1/2,n}}{\Delta x} = \frac{\theta_{m-1,n} - 2\theta_{m,n} + \theta_{m+1,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2\theta}{\partial y^2}\right|_{m,n} = \frac{\theta_{m,n-1} - 2\theta_{m,n} + \theta_{m,n+1}}{(\Delta y)^2}$$

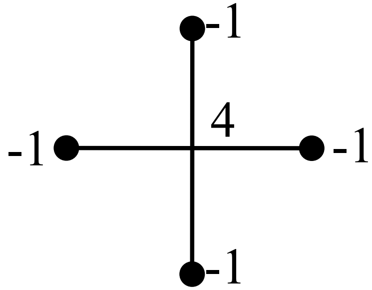
$$\boxed{-(\Delta x)^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) = 4\theta_{m,n} - \theta_{m,n-1} - \theta_{m,n+1} - \theta_{m-1,n} - \theta_{m+1,n}}$$

Conduction bidimensionnelle régime stationnaire

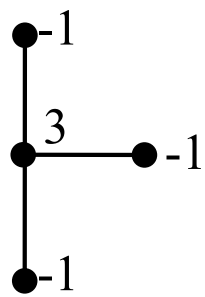
Solution numérique

- **Schémas de calcul**

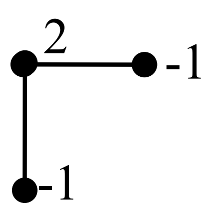
$$-(\Delta x)^2\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)=4\theta_{m,n}-\theta_{m,n-1}-\theta_{m,n+1}-\theta_{m-1,n}-\theta_{m+1,n}$$



$$4\theta_{m,n}-\theta_{m,n-1}-\theta_{m,n+1}-\theta_{m-1,n}-\theta_{m+1,n}$$

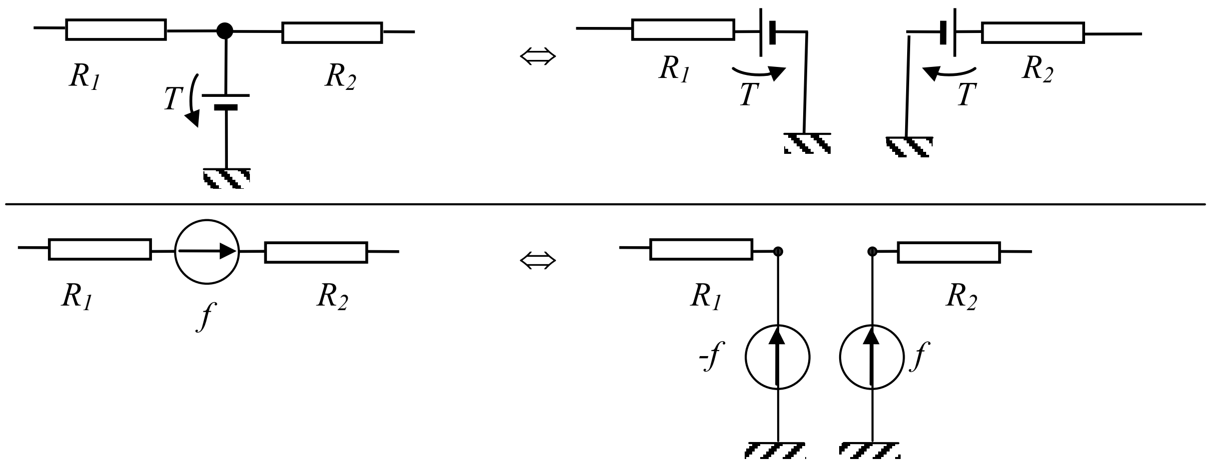


$$3\theta_{m,n}-\theta_{m,n-1}-\theta_{m,n+1}-\theta_{m+1,n}$$

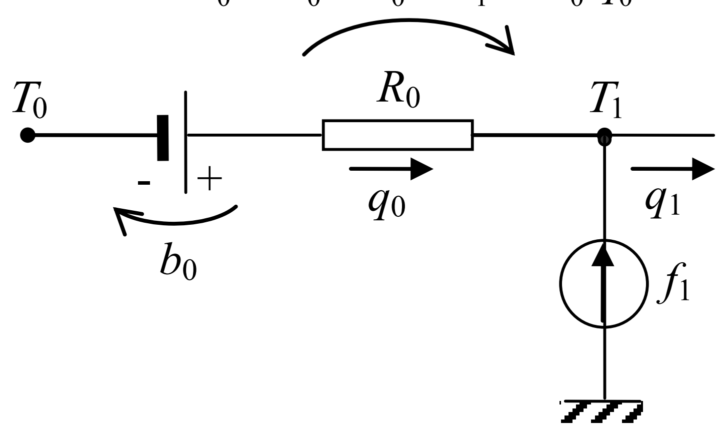


$$2\theta_{m,n}-\theta_{m,n+1}-\theta_{m+1,n}$$

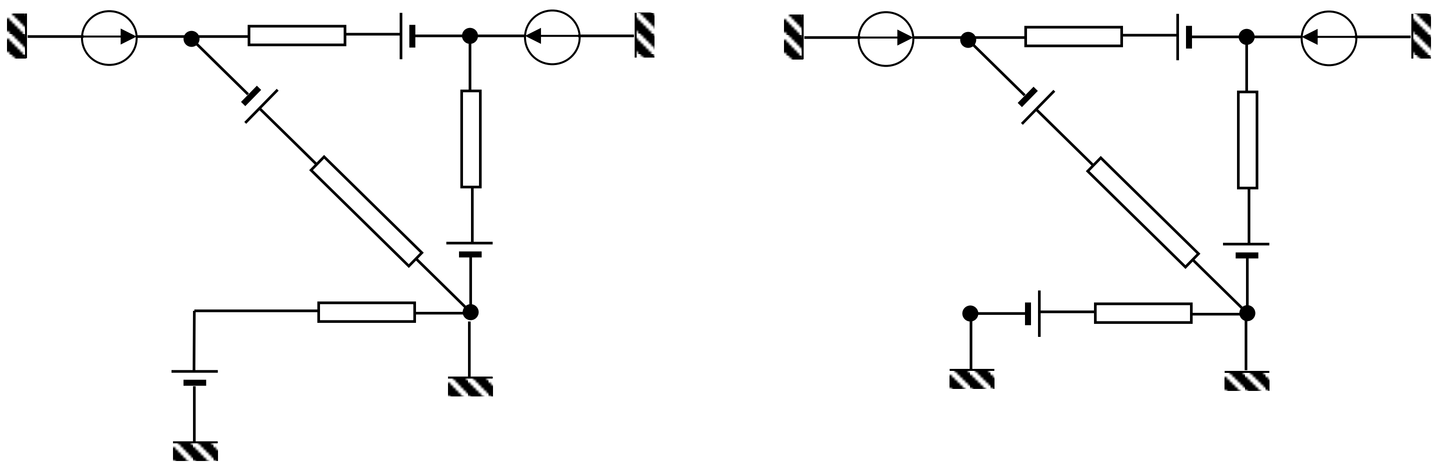
1. Forme discrète de l'équation de Poisson



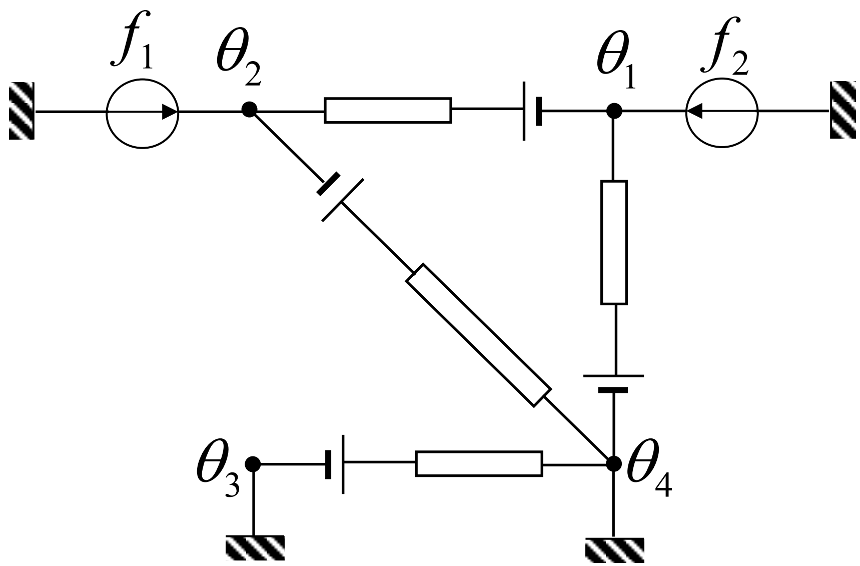
$$e_0 = T_0 + b_0 - T_1 = R_0\, q_0$$



1. Forme discrète de l'équation de Poisson



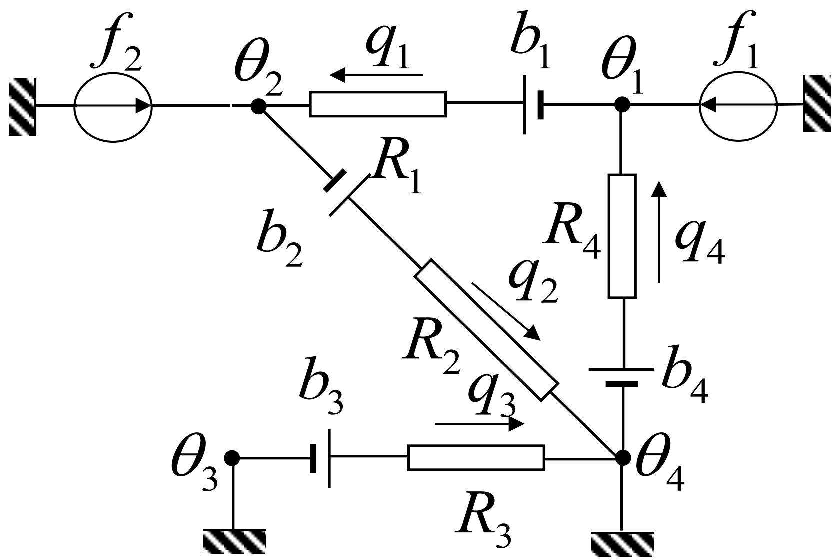
2. Numéroter les nœuds et les sources de flux

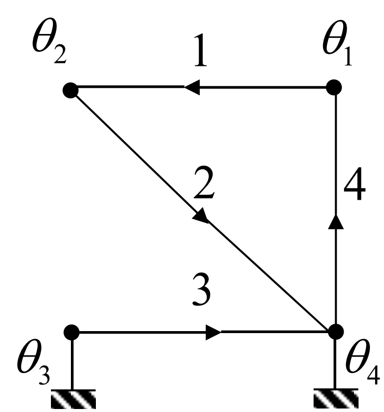


Conduction

- Conduction - général
  - Loi Fourier
  - Equation chaleur
  - Formes particulaires
  - Conditions limites
- Conduction 2D
  - Sans sources
  - Résistance therm.
  - Avec sources
  - Circuits thermiques
  - Neumann

3. Numéroter les branches et assigner un sens

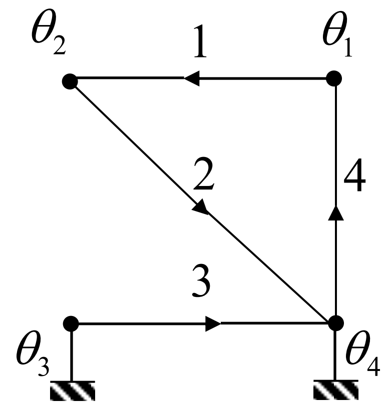




Conduction bidimensionnelle régime stationnaire

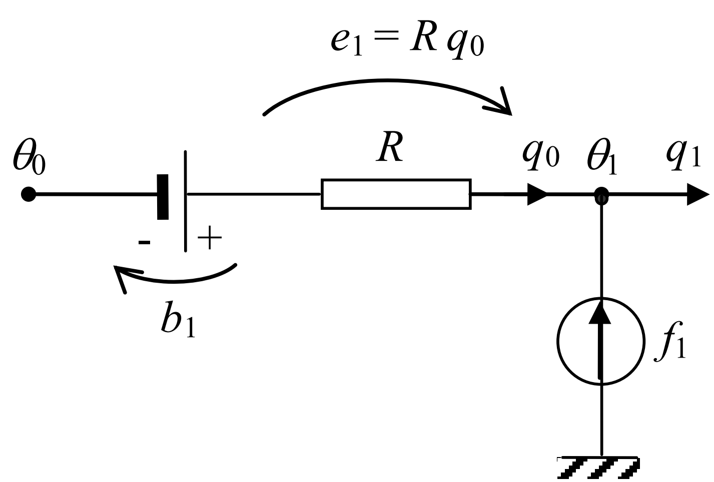
Solution numérique

4. Matrice des connexions et d'incidence



$$
\begin{array}{ccccccc}
 & \text{nœud} & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \text{arc} \\
\mathbf{A}_0 = & \left[\begin{array}{cccc} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right] & & & & & \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}
\end{array}
$$

$$
\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}
$$



$$
\boldsymbol{\theta} = (\mathbf{A}^T\mathbf{G}\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{G}\mathbf{b} + \mathbf{f})
$$

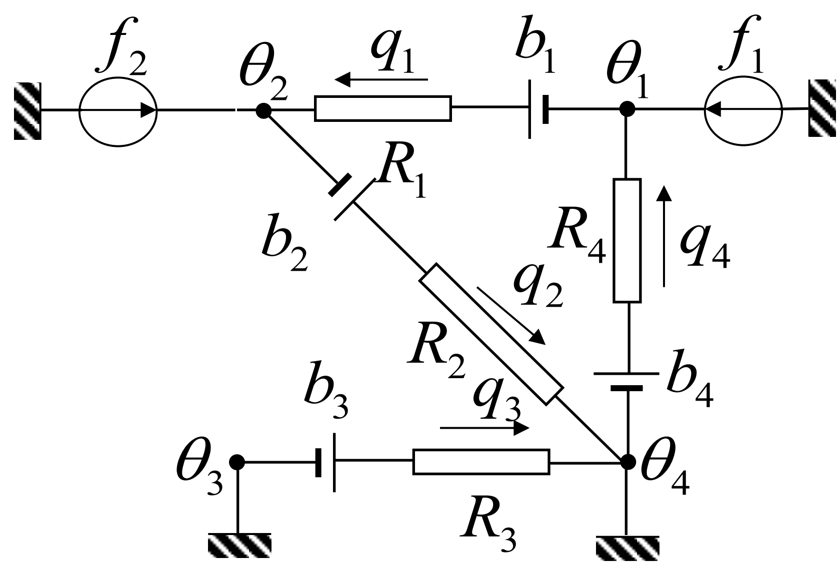
5. Matrices, vecteurs et solution

nœud

arc

$$
\begin{array}{c c}
 & \begin{array}{cc} \theta_1 & \theta_2 \end{array} \\
\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} & \mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{array}
\qquad
\mathbf{G} = \begin{bmatrix} 1/R_1 & 0 & 0 & 0 \\ 0 & 1/R_2 & 0 & 0 \\ 0 & 0 & 1/R_3 & 0 \\ 0 & 0 & 0 & 1/R_4 \end{bmatrix}
\qquad
\mathbf{b} = \begin{bmatrix} T_1 \\ -T_2 \\ T_3 \\ T_4 \end{bmatrix}
$$

$$\mathbf{f} = [f_1 \quad f_2]^T$$



$$\mathbf{A}^T\mathbf{G}\mathbf{A}\boldsymbol{\theta} = (\mathbf{A}^T\mathbf{G}\mathbf{b} + \mathbf{f})$$

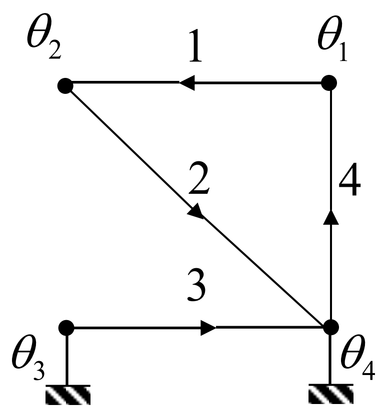
$$\boldsymbol{\theta} = (\mathbf{A}^T\mathbf{G}\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{G}\mathbf{b} + \mathbf{f})$$

Conduction bidimensionnelle régime stationnaire

Solution numérique

Conduction

- Conduction - général
  - Loi Fourier
  - Equation chaleur
  - Formes particulaires
  - Conditions limites
- **Conduction 2D**
  - Sans sources
  - Résistance therm.
  - Avec sources
  - Circuits thermiques
  - **Neumann**



Matrice des connexions

$$\mathbf{A}_0 = \begin{array}{cc} \text{nœud} \quad \begin{array}{cccc} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{array} & \text{arc} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} & \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \end{array}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{\theta} = (\mathbf{A}^T \mathbf{G} \mathbf{A})^{-1} (\mathbf{A}^T \mathbf{G} \mathbf{b} + \mathbf{f})$$

Conduction bidimensionnelle régime stationnaire

Solution numérique

$$\begin{cases} e_1 = b_1 + \theta_1 - \theta_2 \\ e_2 = b_2 + \theta_2 \\ e_3 = b_3 \\ e_4 = b_4 + \theta_4 - \theta_1 \end{cases} \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\mathbf{e} = \mathbf{b} - \mathbf{A\theta}$$

$$\mathbf{q} = \mathbf{Ge}$$

---

nœud

$$\begin{array}{c} 1 \\ 2 \end{array} \begin{cases} -q_1 + q_4 = -f_1 \\ q_1 - q_2 = -f_2 \end{cases} \qquad \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = -\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$-\mathbf{A}^T\mathbf{q} = \mathbf{f}$$

$$\mathbf{f} = -\mathbf{A}^T\mathbf{q} = -\mathbf{A}^T\mathbf{Ce} = -\mathbf{A}^T\mathbf{C}(\mathbf{b} - \mathbf{A\theta})$$

$$\mathbf{\theta} = (\mathbf{A}^T\mathbf{CA})^{-1}(\mathbf{A}^T\mathbf{Cb} + \mathbf{f})$$

Conduction

Sol. numérique 1D
- Loi Fourier
- Dirichlet
- Formes particulaires
- Conditions limites

Conduction 1D
- Sans sources
- Résistance therm.
- Avec sources
- Circuits thermiques

- **Analyse nodale des circuits**

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\boldsymbol{\theta} \arrow[d, "\text{Connexions}"', "-\mathbf{A}"] & \mathbf{f} \\
\mathbf{e}-\mathbf{b} \arrow[r, "\text{Loi constitutive}", "\mathbf{C}"'] & \mathbf{q} \arrow[u, "\text{Bilans}", "-\mathbf{A}^T"']
\end{tikzcd}
$$

$\boldsymbol{\theta}$	températures dans les nœuds
$\mathbf{e}$	chute de température sur les résistances
$\mathbf{q}$	flux dans les résistances
$\mathbf{f}$	flux externes