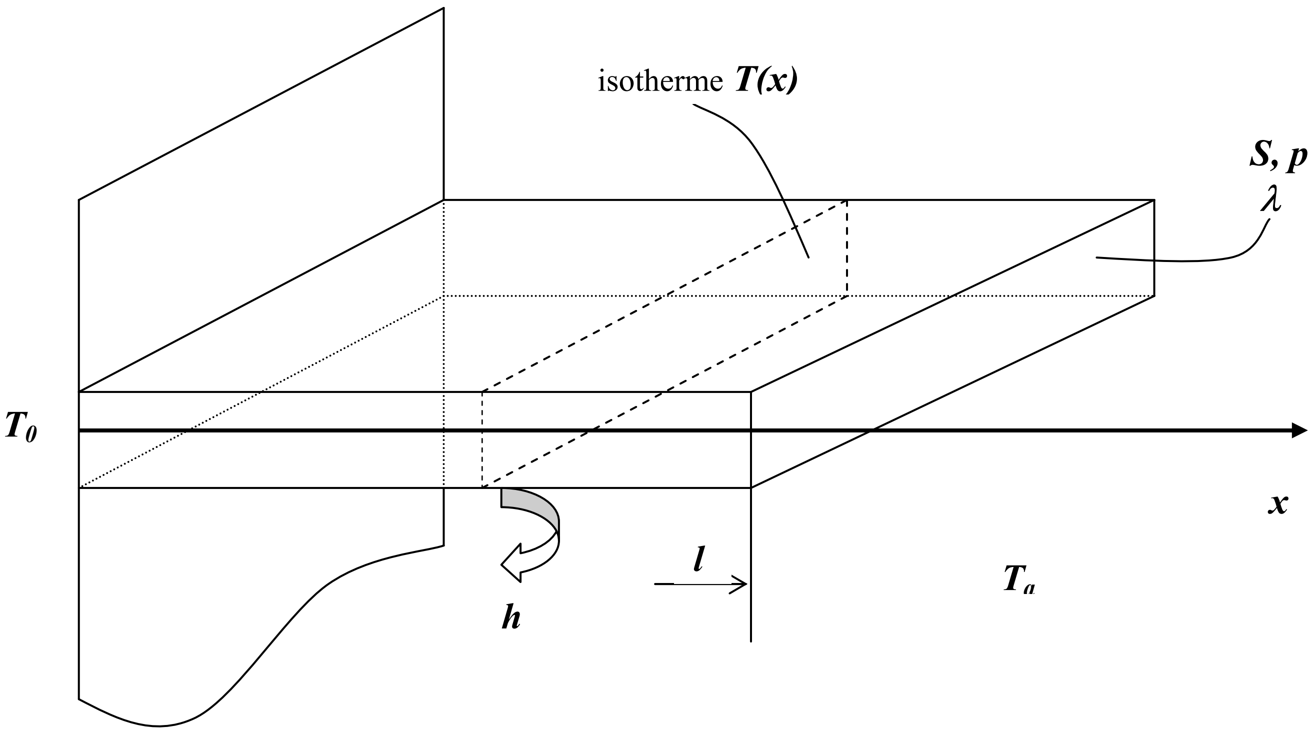
AILETTE DE REFROIDISSEMENT

Soit une ailette de refroidissement de section droite, uniformément chauffée à sa base à la température T_0 et placée dans un milieu ambiant à la température T_a.

On suppose que la température est uniquement fonction de x, donc les isothermes sont planes et perpendiculaires à l'axe x.

On note p le périmètre de base, S la section droite, λ la conductivité thermique et h le coefficient d'échange superficiel de l'ailette.



1). L'ailette est de longueur infinie. Calculer $T = f(x)$ et son efficacité :

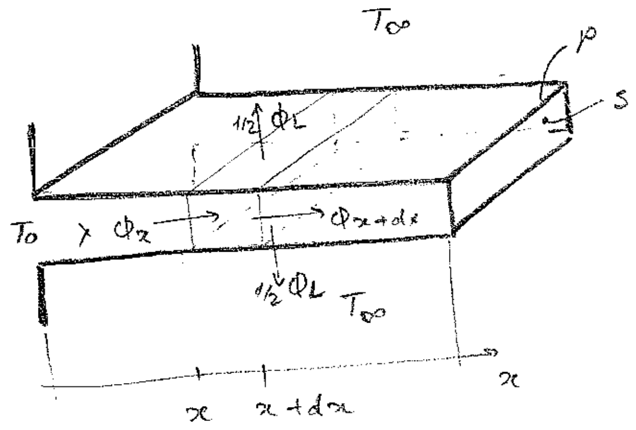
$$E = \frac{\Phi_0}{\Phi_0'}$$

Φ_0 flux évacué au pied de l'ailette,

Φ_0' flux évacué par la section S s'il n'y avait pas d'ailette ;

2). L'ailette est de longueur l finie et isolée (flux nul) en $x = l$. Mêmes questions.

3). L'ailette est de longueur finie avec un échange superficiel en $x = l$. Mêmes questions



Bilan pour le volume $S \cdot dx$

$$\underbrace{\Phi_x}_{\text{entre par conduction}} - \underbrace{\Phi_{x+dx}}_{\text{sort par conduction}} - \underbrace{\Phi_L}_{\text{latéral, par convection}} = 0$$

$$\Phi_x = -\lambda\, \text{grad}\, T \cdot S(x) = -\lambda \left(\frac{dT}{dx}\right)_x \cdot S(x)$$

$$\Phi_{x+dx} = -\lambda \left(\frac{dT}{dx}\right)_{x+dx} S(x)$$

$$\Phi_L = h\,(T_x - T_\infty)\, P(x) \cdot dx$$

$$-\lambda \left[\left(\frac{dT}{dx}\right)_x - \left(\frac{dT}{dx}\right)_{x+dx}\right] S(x) - h\,P(x)\,(T_x - T_\infty)\,dx = 0$$

$$\theta \equiv T - T_\infty \implies \frac{dT}{dx} = \frac{d\theta}{dx}$$

$$\lambda \frac{d^2\theta}{dx^2} S(x) = h\,P(x) \cdot \theta \qquad \alpha \equiv \frac{h}{\lambda}\frac{P}{S}$$

$$\boxed{\frac{d^2\theta}{dx^2} = \alpha\,\theta} \iff \boxed{\Delta\theta = \alpha\,\theta} \iff \boxed{\text{div}(\overrightarrow{\text{grad}}\,\theta) = \alpha\,\theta}$$

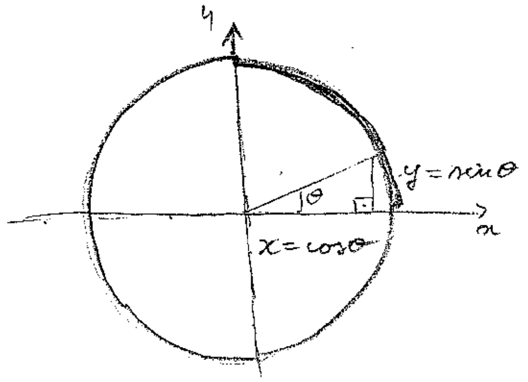
$$\frac{d}{dx} \equiv D \qquad \alpha \equiv \omega^2 \implies \omega = \sqrt{\alpha}$$

$$D^2\theta = \omega^2\theta \implies (D^2 - \omega^2)\,\theta = 0 \implies (D-\omega)(D+\omega)\,\theta = 0$$

solutions: $(D-\omega)\,\theta = 0 \implies D\theta = \omega\theta \implies \theta_1 = A_1 e^{\omega x}$

$(D+\omega)\,\theta = 0 \implies D\theta = -\omega\theta \implies \theta_2 = A_2 e^{-\omega x}$

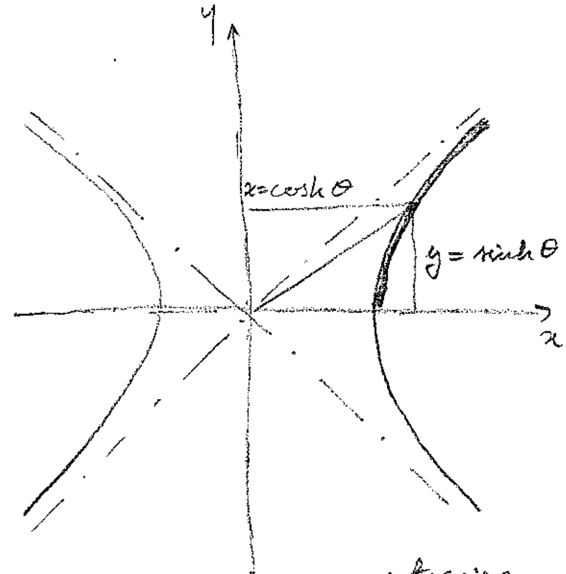
$$\boxed{\theta = A_1 e^{\omega x} + A_2 e^{-\omega x}}$$



cercle unitaire

$x^2 + y^2 = 1$

$\cos^2\theta + \sin^2\theta = 1$



hyperbole unitaire

$x^2 - y^2 = 1$

$\cosh^2\theta - \sinh^2\theta = 1$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1}$$

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$e^x = \cosh x + \sinh x$

$e^{-x} = \cosh x - \sinh x$

$$\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\sinh ix = \frac{e^{ix} - e^{-ix}}{2} = i \sin x$$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\theta = A_1 e^{-\omega x} + A_2 e^{\omega x}$ Équation générale

Ailette infinie

$x \to \infty \Rightarrow T(x) \to T_\infty \Rightarrow \theta(x) = T(x) - T_\infty = 0$

$x = 0 \Rightarrow T(x) = T_0 \Rightarrow \theta(0) = T_0 - T_\infty$

$\theta(\infty) = 0 \Rightarrow A_1 e^{-\omega\infty} + A_2 e^{\omega\infty} = 0 \Rightarrow A_2 = 0$

$\theta(0) = \theta_0 \Rightarrow A_1 \cdot e^0 = \theta_0 \Rightarrow A_1 = T_0 - T_\infty$

$$\boxed{\theta = \theta_0 e^{-\omega x}}$$

$T - T_\infty = (T_0 - T_\infty) e^{-\sqrt{\alpha}\cdot x}$

$$\boxed{T = T_\infty + (T_0 - T_\infty) e^{-\sqrt{\alpha} x}}$$

Efficacité

$\phi_0 = \int_0^\infty h p (T_x - T_\infty) dx$ flux convectif total

$\phi_0 = -\lambda \left(\frac{dT}{dx}\right)_{x=0} \cdot S$ flux par la surface S dans la section $x=0$

$\frac{dT}{dx} = -(T_0 - T_\infty)\sqrt{\alpha}\, e^{-\sqrt{\alpha} x}$ $(e^f)' = f' e^f$

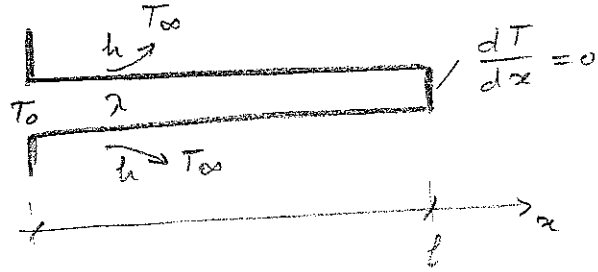
$\left.\frac{dT}{dx}\right|_{x=0} = -(T_0 - T_\infty)\sqrt{\alpha}$

$\phi_0 = +\lambda (T_0 - T_\infty)\sqrt{\alpha} \cdot S$

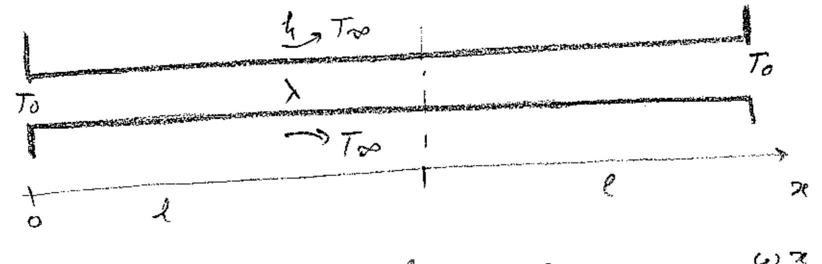
$\phi_0' = h S (T_0 - T_\infty)$

$$\boxed{E = \frac{\phi_0}{\phi_0'} = \frac{\lambda\sqrt{\alpha}}{h}}$$

$$\boxed{\text{Ailette de longueur } l \text{ isolée (flux nul) en } x = l}$$



Note : $\frac{dT}{dx} = 0 \rightarrow$ flux négligeable ou ailette symétrique



Equation générale : $\theta(x) = A_1 e^{\omega x} + A_2 e^{-\omega x}$; $\theta = T_0 - T_\infty$

$\omega = \sqrt{\alpha} = \sqrt{\frac{hp}{\lambda S}}$

Conditions aux limites :

$x = 0 \Longrightarrow \theta(0) = \theta_0 \Longrightarrow \theta_0 = A_1 + A_2$

$x = l \Longrightarrow \left.\frac{d\theta}{dx}\right|_{x=l} = 0 \Longrightarrow \left(\cancel{\omega} A_1 e^{\omega x} - \cancel{\omega} A_2 e^{-\omega x}\right)_{x=l} = 0$

$$\begin{cases} A_1 e^{\omega l} - A_2 e^{-\omega l} = 0 \\ A_1 + A_2 = \theta_0 \end{cases} \qquad \begin{array}{l} (1) \\ \times e^{-\omega l} \end{array} \qquad \begin{array}{l} (2) \\ \times (-e^{\omega l}) \end{array}$$

(1) $A_1 (e^{\omega l} + e^{-\omega l}) = \theta_0 e^{-\omega l} \Longrightarrow A_1 = \frac{\theta_0}{e^{\omega l} + e^{-\omega l}} \cdot e^{-\omega l} = \frac{\theta_0}{2 \cosh(\omega l)} \cdot e^{-\omega l}$

(2) $-A_2 (e^{-\omega l} + e^{\omega l}) = -\theta_0 e^{\omega l} \Longrightarrow A_2 = \frac{\theta_0}{e^{\omega l} + e^{-\omega l}} e^{\omega l} = \frac{\theta_0}{2 \cosh(\omega l)} \cdot e^{\omega l}$

$$\theta(x) = A_1 e^{\omega x} + A_2 e^{-\omega x}$$

$$= \frac{\theta_0}{2 \cosh(\omega l)} \left(e^{-\omega l} \cdot e^{\omega x} + e^{\omega l} \cdot e^{-\omega x}\right)$$

$$= \frac{\theta_0}{2 \cosh(\omega l)} \left(e^{-\omega (l - x)} + e^{\omega (l - x)}\right)$$

$$\boxed{\theta(x) = \theta_0 \, \frac{\cosh(\omega(l - x))}{\cosh(\omega l)}}$$

Efficacité de l'ailette : $E = \dfrac{\Phi_0}{\Phi_0'}$

$$\Phi_0 = -\lambda \left(\frac{dT}{dx}\right)_{x=0} \cdot S = -\lambda \left(\frac{d\theta}{dx}\right)_{x=0} \cdot S$$

$$= -\lambda S \cdot \frac{\theta_0}{\cosh(\omega l)} \cdot \left(\frac{d}{dx} \cosh(\omega(l-x))\right)_{x=0}$$

$$= -\lambda S \cdot \frac{\theta_0}{\cosh(\omega l)} \cdot \Big[-\omega \sinh(\omega(l-x))\Big]_{x=0}$$

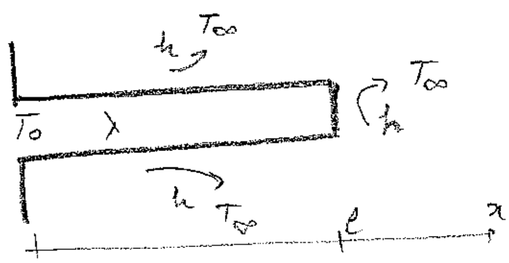
$$= \lambda \omega S \, \theta_0 \frac{\sinh(\omega l)}{\cosh(\omega l)} \qquad \lambda \omega S = \lambda S \cdot \sqrt{\frac{hp}{\lambda S}} = \sqrt{hp\lambda S}$$

$$\Phi_0 = \sqrt{h\lambda p S}\ \theta_0 \tanh(\omega l)$$

$$\Phi_0' = hS(T_0 - T_\infty) = hS\theta_0$$

$$\boxed{E = \frac{\Phi_0}{\Phi_0'} = \frac{\sqrt{hS\lambda p}}{hS} \tanh(\omega l) = \sqrt{\frac{\lambda p}{hS}} \tanh(\omega l)}$$

$\boxed{\text{Echange superficielle en } x = l}$



Equation générale:

$$\theta = A_1 e^{\omega x} + A_2 e^{-\omega x} \; ; \; \theta \equiv \theta(x)$$

Conditions aux limites

$$x = 0 \Longrightarrow \theta = \theta_0 = T_0 - T_\infty \qquad \Longrightarrow \theta_0 = A_1 + A_2$$

$$x = l \Longrightarrow \; -\lambda \left.\frac{d\theta}{dx}\right|_{x=l} = h \cdot \theta\Big|_{x=l} \qquad \theta_l \equiv T_l - T_\infty$$

$$\left.\frac{d\theta}{dx}\right|_{x=l} = -\frac{h}{\lambda}\theta_l$$

$$\omega A_1 e^{\omega l} - \omega A_2 e^{-\omega l} = -\frac{h}{\lambda}\left(A_1 e^{\omega l} + A_2 e^{-\omega l}\right)$$

$$\begin{cases} A_1\left(\omega + \frac{h}{\lambda}\right)e^{\omega l} - A_2\left(\omega - \frac{h}{\lambda}\right)e^{-\omega l} = 0 \\ A_1 + A_2 = \theta_0 \quad \Big| \times\left(\omega - \frac{h}{\lambda}\right)e^{-\omega l} \Big| \end{cases}$$

$$\times\left[-\left(\omega + \frac{h}{\lambda}\right)e^{\omega l}\right]$$

$$A_1\left[\omega \cdot 2ch(\omega l) + \frac{h}{\lambda} 2sh(\omega l)\right] = \theta_0 \left(\omega - \frac{h}{\lambda}\right)e^{-\omega l}$$

$$+A_2\left[\omega \cdot 2ch(\omega l) + \frac{h}{\lambda} 2sh(\omega l)\right] = +\theta_0 \left(\omega + \frac{h}{\lambda}\right)e^{\omega l}$$

$$\theta = A_1 e^{\omega x} + A_2 e^{-\omega x} =$$

$$= \frac{\theta_0}{2\left[\omega\, ch(\omega l) + \frac{h}{\lambda} sh(\omega l)\right]} \cdot \left[\omega \underbrace{\left(e^{-\omega l} \cdot e^{\omega x} + e^{\omega l} e^{-\omega x}\right)}_{\underbrace{\left(e^{-\omega(l-x)} + e^{\omega(l-x)}\right)}_{2ch(\omega(l-x))}} + \frac{h}{\lambda}\underbrace{\left(-e^{-\omega l} e^{\omega x} + e^{\omega l} e^{-\omega x}\right)}_{\underbrace{\left(-e^{-\omega(l-x)} + e^{\omega(l-x)}\right)}_{2sh(\omega(l-x))}}\right]$$

$$= \theta_0 \frac{\omega\, ch(\omega(l-x)) + \frac{h}{\lambda} sh(\omega(l-x))}{\omega\, ch(\omega l) + \frac{h}{\lambda} sh(\omega l)}$$

$$\boxed{\theta = \theta_0 \frac{ch(\omega(l-x)) + \frac{h}{\lambda\omega} sh(\omega(l-x))}{ch(\omega l) + \frac{h}{\lambda\omega} sh(\omega l)}}$$

Efficacité de l'ailette : $E = \dfrac{\Phi_0}{\Phi_0'}$

$$\Phi_0 = -\lambda \left(\frac{dT}{dx}\right)_{x=0} \cdot S = -\lambda \left(\frac{d\theta}{dx}\right)_{x=0} \cdot S$$

$$\left(\frac{d}{dx} ch(\omega(l-x))\right)\Big|_{x=0} = \left(-\omega\, sh(\omega(l-x))\right)\Big|_{x=0} = -\omega\, sh(\omega l)$$

$$\frac{d}{dx} sh(\omega(l-x))\Big|_{x=0} = -\omega\, ch(\omega(l-x))\Big|_{x=0} = -\omega\, ch(\omega l)$$

$$\Phi_0 = \omega \lambda S \theta_0 \frac{sh(\omega l) + \frac{h}{\lambda\omega} ch(\omega l)}{ch(\omega l) + \frac{h}{\lambda\omega} sh(\omega l)}$$

$$\Phi_0 = \omega \lambda S \theta_0 \frac{th(\omega l) + \frac{h}{\lambda\omega}}{1 + \frac{h}{\lambda\omega} th(\omega l)}$$

$$\Phi_0' = h S \theta_0$$

$$\boxed{E = \frac{\Phi}{\Phi_0'} = \frac{\lambda\omega}{h} \cdot \frac{th(\omega l) + \frac{h}{\lambda\omega}}{1 + \frac{h}{\lambda\omega} th(\omega l)} = \frac{1 + \frac{\lambda\omega}{h} th(\omega l)}{1 + \frac{h}{\lambda\omega} th(\omega l)}}$$