INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

Building design: multidisciplinary approach

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membre de UNIVERSITÉ DE LYON

Curricula

Lectures

- 2h Introduction
- 2 x 2h Geotechnics
- 2 x 2h Structures
- 2 x 2h HVAC systems
- 2 x 2h Hydraulic systems

Tutorials

- 2h Geotechnics
- 2h Structures
- 2h HVAC systems
- 2h Hydraulic systems

Project

- 5 x 2h Geotechnics
- 5 x 2h Structures
- 5 x 2h HVAC systems
- 5 x 2h Hydraulic systems

Evaluation

- 1/3 continuous monitoring (indiv.)
- 1/3 written report of the project (coll.)
- 1/3 project oral defense (indiv..)

Applied mathematics

1. **Constructing the model**: steady-state and dynamic
2. **Solving**: steady state and time-dependent matrix and differential equations.

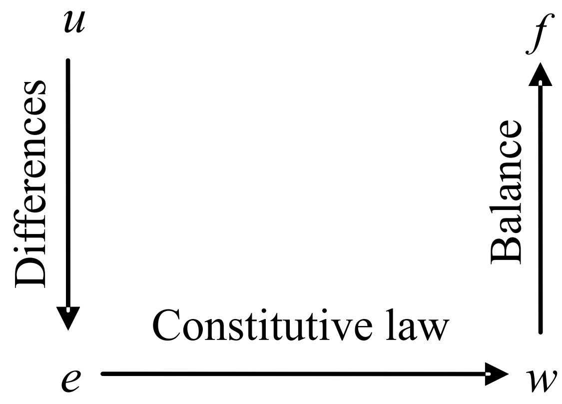
4 simplifications:

1. linear,
2. discrete,
3. one-dimensional,
4. with constant coefficients.

4 simplifications

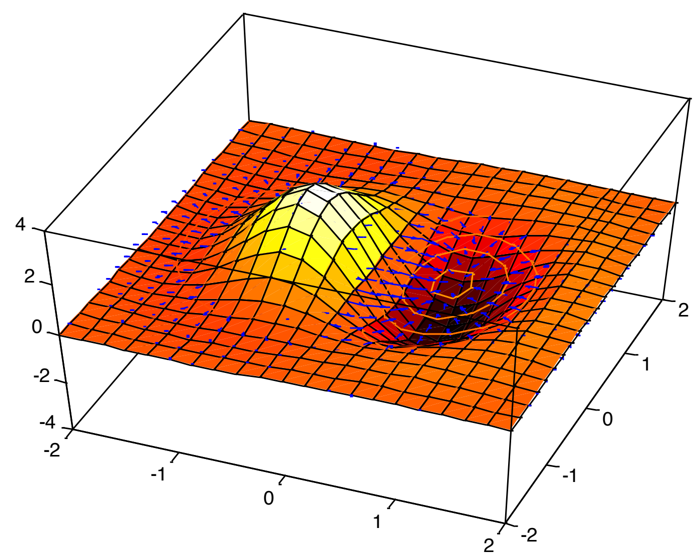
1. **Nonlinear becomes linear:**
   - Hooke’s law in mechanics: displacement is proportional to force
   - Fourier’s law in heat networks: heat flow rate is proportional to temperature difference
   - Bernoulli’s law in hydraulic networks: volumetric flow rate is proportional to linearization of the square root of the pressure difference

Modelling:
structural, thermal, hydraulic, electrical



u	potential
e	potential difference
w	flow rate
f	external flow rate

- Structural mechanics
- Heat transfer
- Hydraulics



Transfer: irreversible statistical phenomena

Space inhomogeneity of an intensive quantity → transport of a physical quantity

Modelling: structural, thermal, hydraulic, electrical

Domain	**Across variable (potential)**	**Through variable (flow)**	**Constitutive equation**	**Balance equation**
Structural mechanics	Displacement u [m]	Internal force v [N]	Hook law $v = -k\, \Delta u$	Force balance $m\ddot{u} = \sum -k\, \Delta u + f$
Heat transfer	Temperature θ [°C]	Heat rate q [W]	Fourier law $q = R^{-1}\, \Delta\theta$	Heat balance $mc\dot{\theta} = \sum R^{-1}\, \Delta\theta + f$
Hydraulics	Pressure p [Pa]	Volumetric flow q [m$^3/s$]	Bernoulli law $q = R^{-1}\sqrt{\Delta p}$	Mass balance $0 = \sum R^{-1}\sqrt{\Delta p} + f$
Electricity				

Structural mechanics: statics

$$\mathbf{u} = (u_1, u_2, u_3) = \textbf{displacements of the masses}$$
$$\mathbf{w} = (w_1, w_2, w_3, w_4) \text{ or } (w_1, w_2, w_3) = \textbf{tension in the springs}.$$

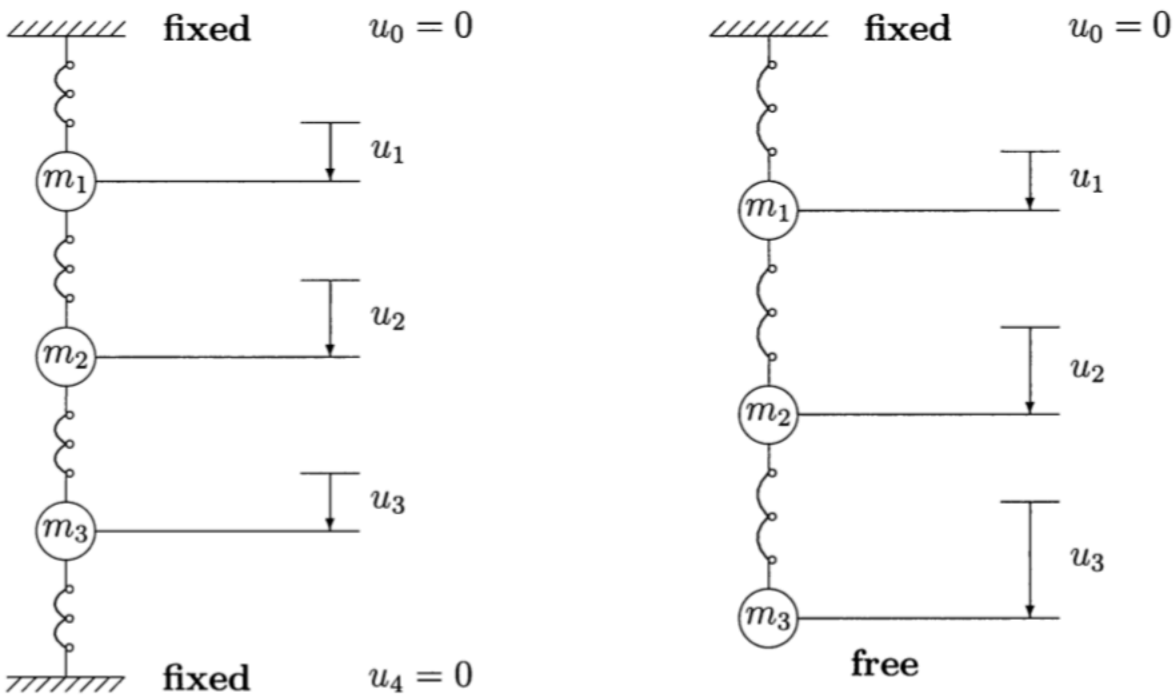
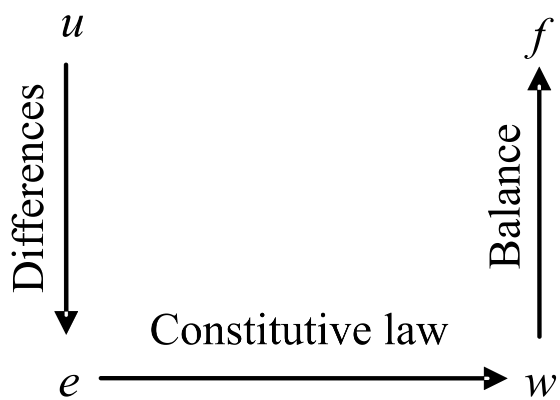


Figure 2.1: Fixed-fixed springs with $u_4 = 0$. Fixed-free with $w_4 = 0$.

Structural mechanics: statics

- 3 steps to create the model



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w	flow rate
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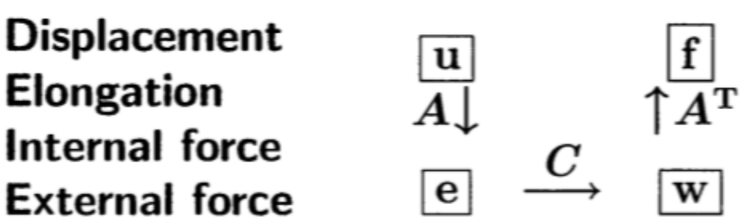
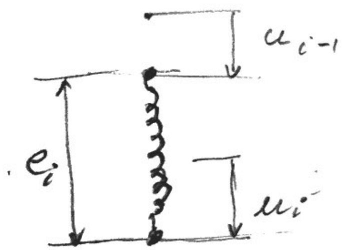
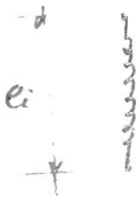
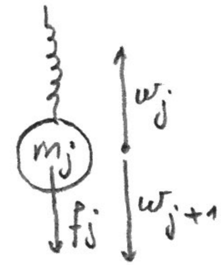


$$\mathbf{e} = A\mathbf{u} \qquad A \text{ is } m \text{ by } n$$
$$\mathbf{w} = C\mathbf{e} \qquad C \text{ is } m \text{ by } m$$
$$\mathbf{f} = A^{\mathrm{T}}\mathbf{w} \qquad A^{\mathrm{T}} \text{ is } n \text{ by } m$$

Structural mechanics: statics



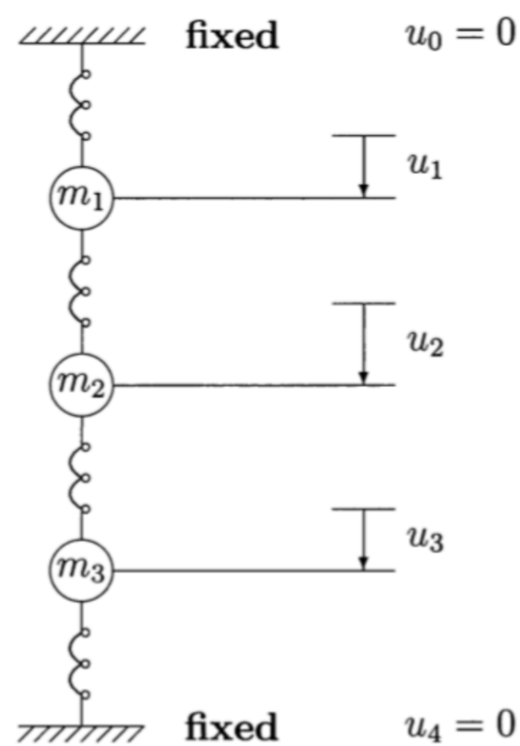




$$e_i = u_i - u_{i-1}$$

$$w_i = c_i e_i$$

$$w_j - w_{j+1} = f_j$$

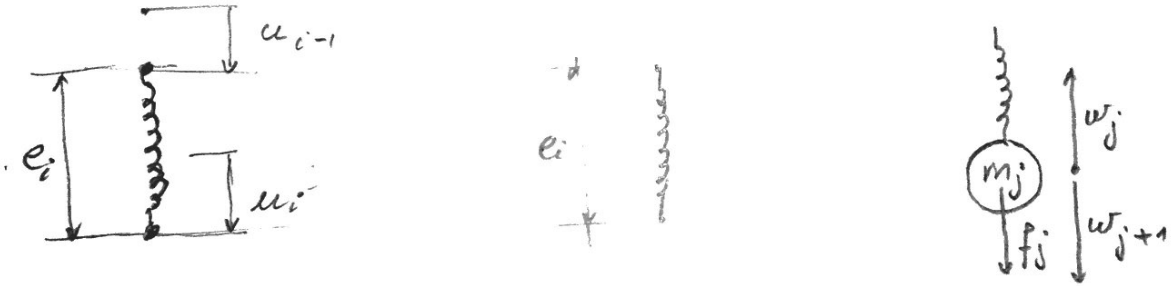


STEP 1: The *elongation* e is the stretching—how far the springs are extended.

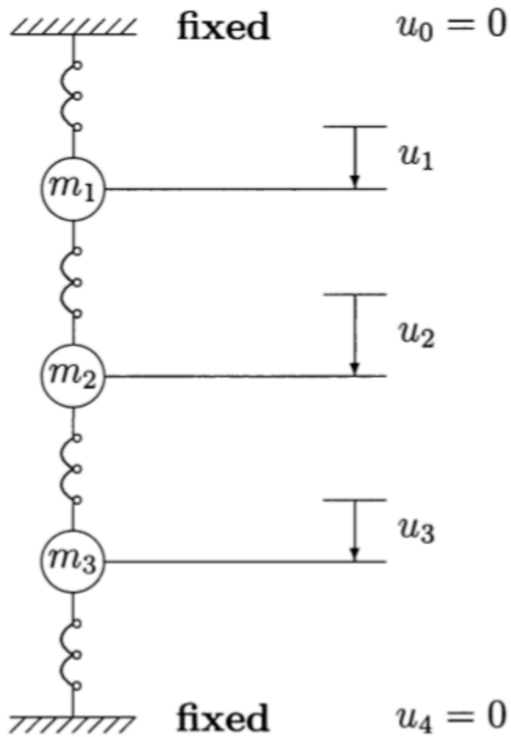
Stretching
$\boldsymbol{e = Au}$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 - 0 \\ u_2 - u_1 \\ u_3 - u_2 \\ 0 - u_3 \end{bmatrix}$$

Structural mechanics: statics



$e_i = u_i - u_{i-1}$ $\qquad$ $w_i = c_i e_i$ $\qquad$ $w_j - w_{j+1} = f_j$

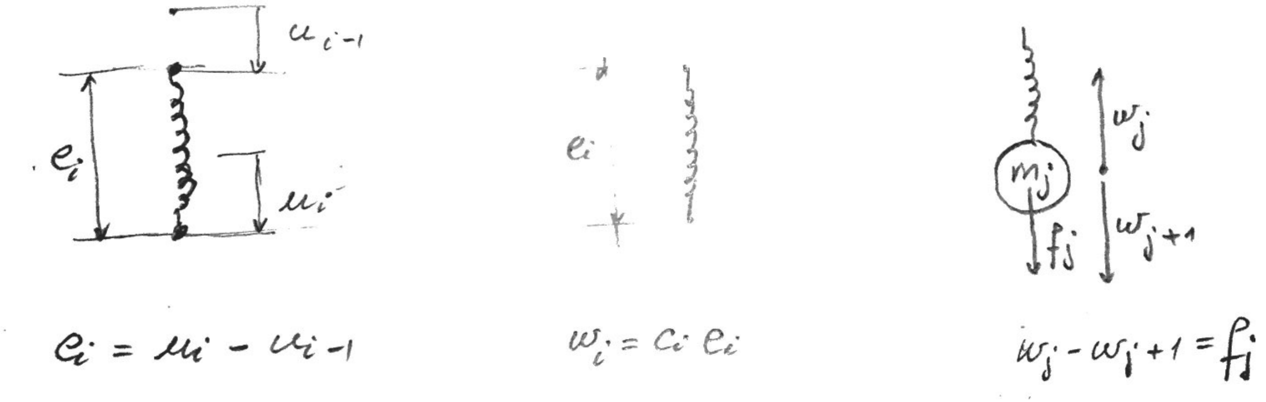


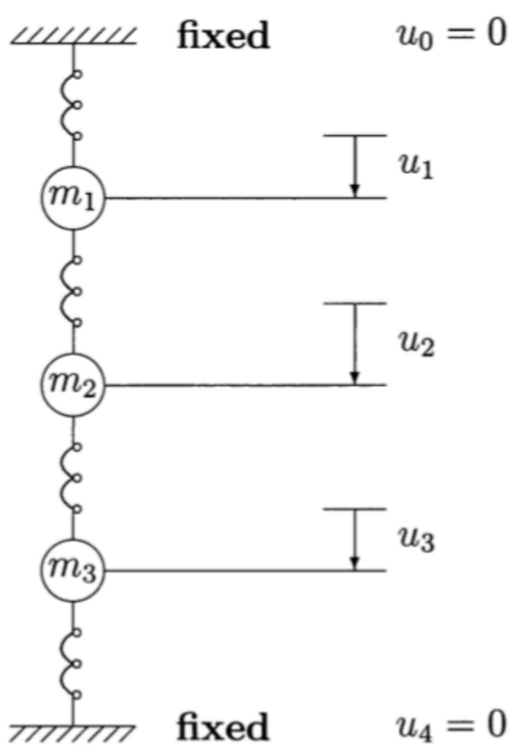
STEP 2: The physical equation $w = Ce$ connects the spring elongations e with the spring tensions w. *This is Hooke's Law* $w_i = c_i e_i$ *for each separate spring.* It is the

Hooke's Law
$\boldsymbol{w = Ce}$

$$\begin{aligned} w_1 &= c_1 e_1 \\ w_2 &= c_2 e_2 \\ w_3 &= c_3 e_3 \\ w_4 &= c_4 e_4 \end{aligned} \quad \text{is} \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} \mathbf{c_1} & & & \\ & \mathbf{c_2} & & \\ & & \mathbf{c_3} & \\ & & & \mathbf{c_4} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = Ce\,.$$

Structural mechanics: statics





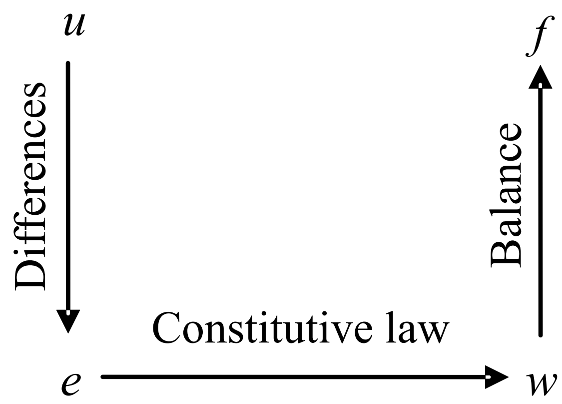
STEP 3: Finally comes the "**balance equation.**"

pulled up by a spring force w_j and down by w_{j+1} plus the gravitational force f_j. Thus $w_j = w_{j+1} + f_j$ or $\boldsymbol{f_j = w_j - w_{j+1}}$:

Balance of forces $\boldsymbol{A^{\mathrm{T}} w = f}$

$$\begin{aligned} f_1 &= w_1 - w_2 \\ f_2 &= w_2 - w_3 \\ f_3 &= w_3 - w_4 \end{aligned} \quad \text{and} \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

Structural mechanics: statics



u	potential
e	potential difference
w	internal (elastic) force
f	external force

$$\begin{matrix} m \text{ by } n \\ m \text{ by } m \\ n \text{ by } m \end{matrix} \quad \left\{ \begin{aligned} \mathbf{e} &= A\mathbf{u} \\ \mathbf{w} &= C\mathbf{e} \\ \mathbf{f} &= A^{\mathrm{T}}\mathbf{w} \end{aligned} \right\} \quad \text{combine into} \quad A^{\mathrm{T}}CA\mathbf{u} = \mathbf{f}\,.$$

stiffness matrix $K = A^T C A$

solution $u = K^{-1} f$

Structural mechanics: minimum principle

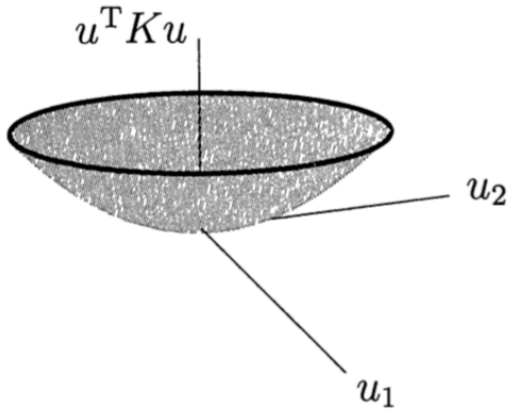
Minimize the total potential energy P $$P(u) = \frac{1}{2}u^{\mathrm{T}}Ku - u^{\mathrm{T}}f\,.$$

The minimum of $P(u) = \frac{1}{2}u^{\mathrm{T}}Ku - u^{\mathrm{T}}f$ **is** $P_{\text{min}} = -\frac{1}{2}f^{\mathrm{T}}K^{-1}f$ **when** $Ku = f$.

Minimize $P(u)$ by solving $\frac{dP}{du} = 0$ or $\mathrm{grad}(P) = 0$ or first variation $\frac{\delta P}{\delta u} = 0$.

$$\frac{1}{2}\boldsymbol{K} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

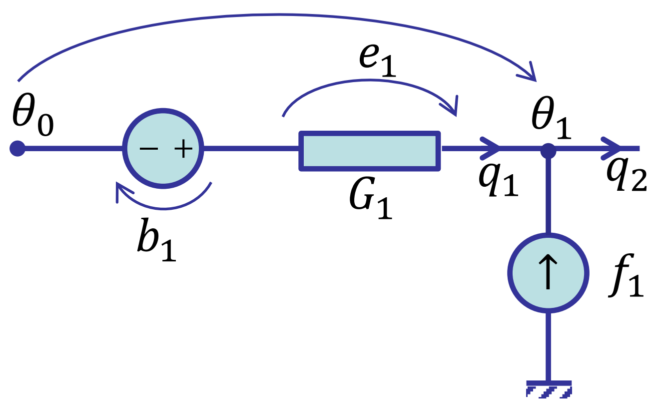
$u_1^2 - u_1u_2 + u_2^2$ $\qquad$ $u^{\mathrm{T}}\left(\frac{1}{2}K\right)u$



Thermal networks

Heat transfer (or heat) is thermal energy in transit due to temperature difference

0th principle : temperature scales ($e_1 = \theta_0 - \theta_1 + b$)
1st principle : energy conservation ($q_1 - q_2 = -f$)
2nd principle and **constitutive laws:** direction / value of heat ($q_1 = G_1 e_1$)



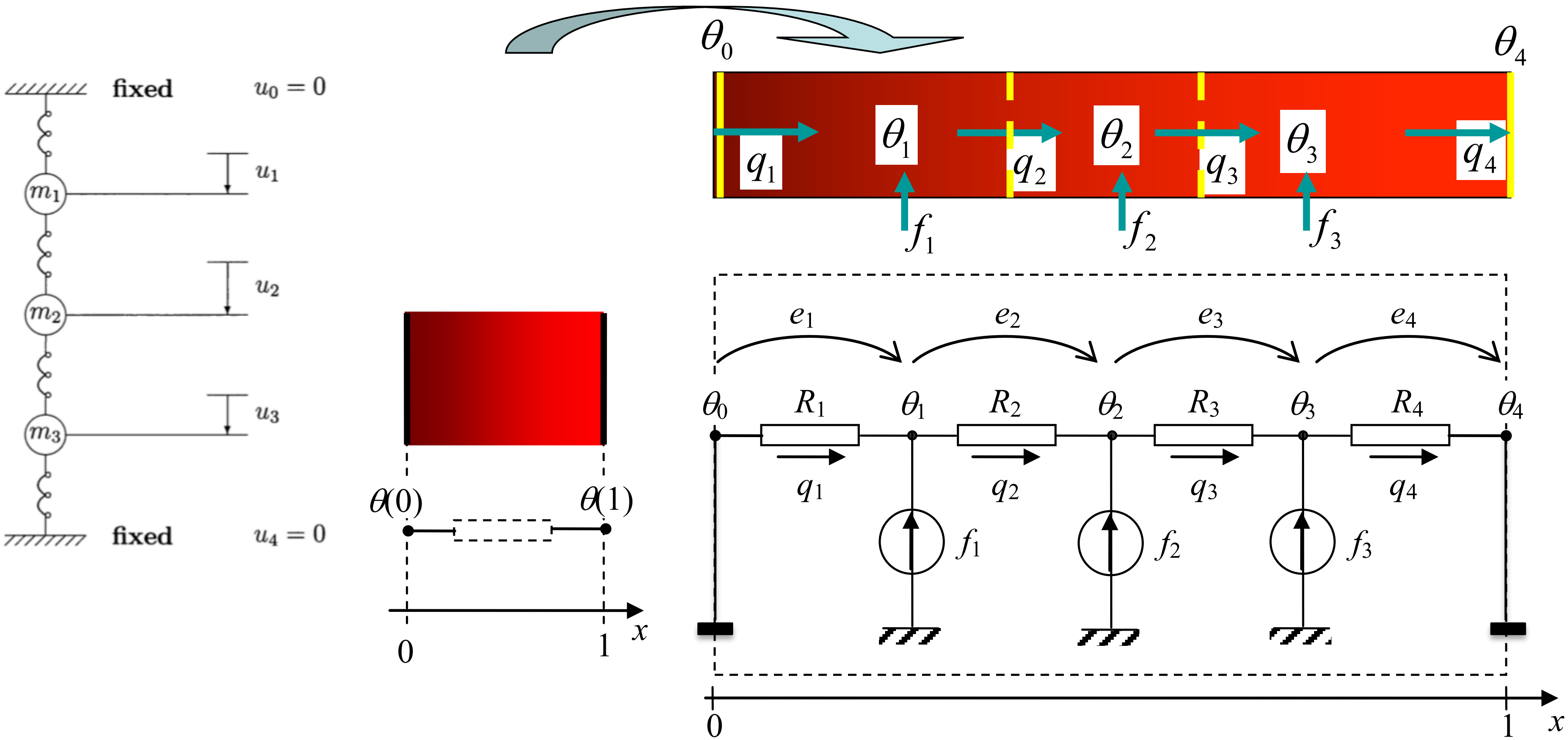
Heat transfer

Heat transfer: thermal energy transport due to temperature difference

Quantity	Meaning	Symbol	Units
Thermal energy	Energy of matter at microscopic level	U	$\mathrm{J} = \mathrm{kg\,m^2 s^{-2}}$
Temperature	Indirect measurement of stored thermal energy	T or θ	K or °C
Heat	Amount of thermal energy transferred	Q	$\mathrm{J} = \mathrm{kg\,m^2 s^{-2}}$
Heat rate	Heat transferred per unit time	$\dot{Q} \equiv q, \Phi$	$\mathrm{W} = \mathrm{kg\,m^2 s^{-3}}$
Heat flux	Heat rate per unit surface area	$\varphi = \frac{dq}{dA}$	$\mathrm{W/m^2} = \mathrm{kg\,s^{-3}}$

Thermal networks: steady-state

$$\mathrm{div}(-\lambda\,\mathbf{grad}\,\theta) = p$$

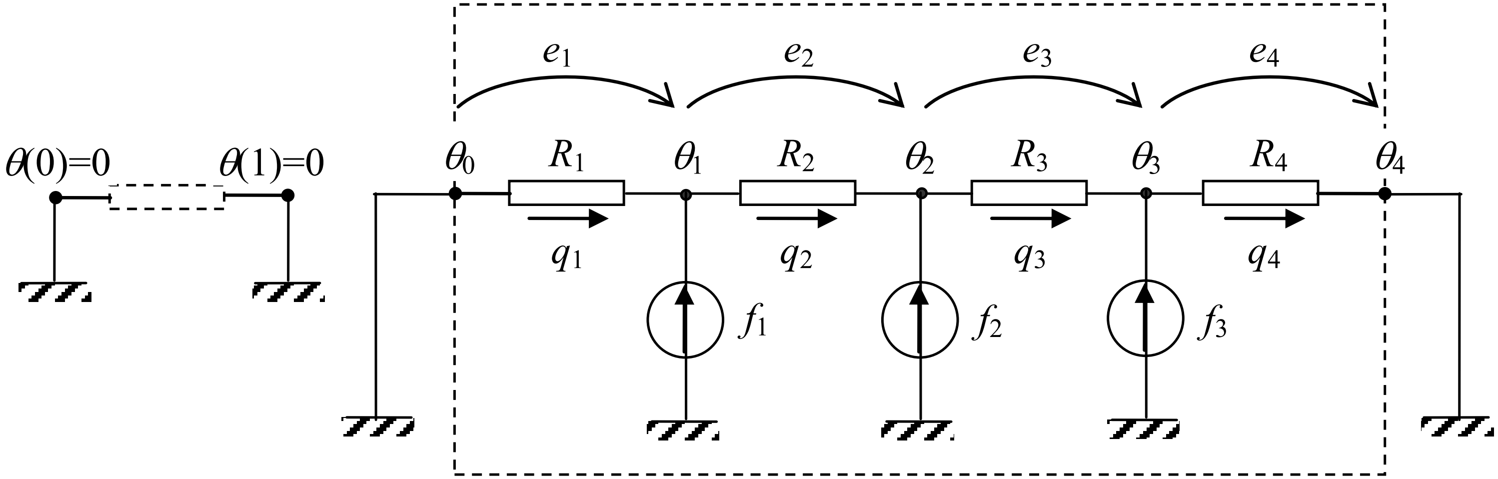


$$\mathrm{div}(-\lambda\,\mathbf{grad}\,\theta) = p \qquad \mathbf{A}^T\mathbf{G}\mathbf{A}\boldsymbol{\theta} = \mathbf{f} \qquad \mathbf{K}\cdot\boldsymbol{\theta} = \mathbf{f}$$

(*a*) (*b*)

Thermal networks: steady-state

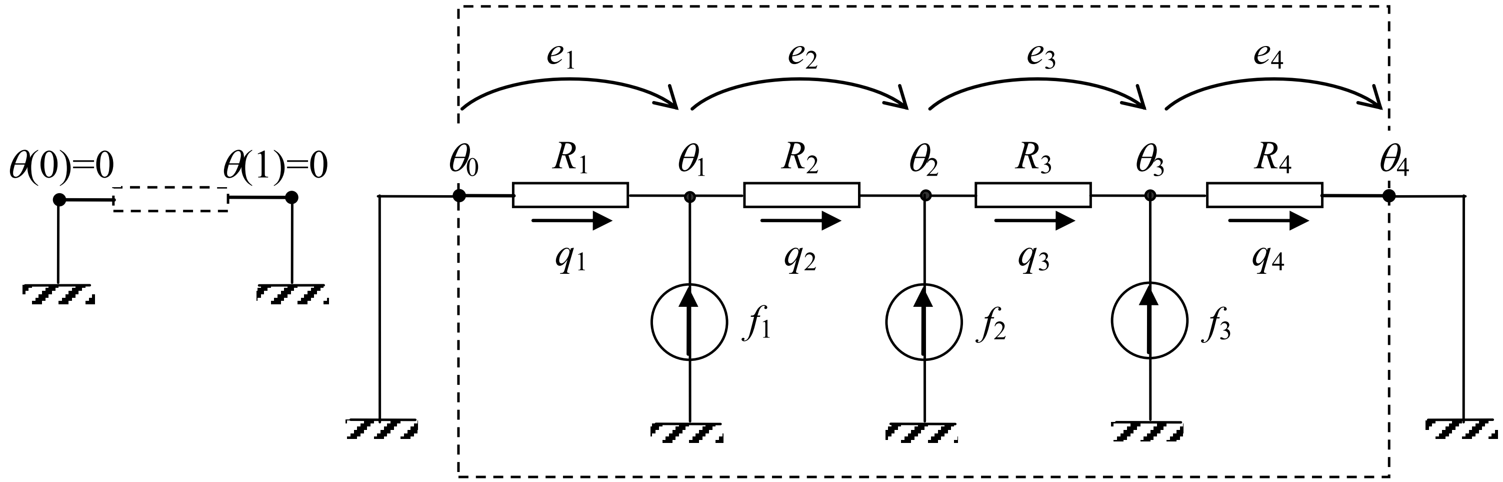
- **Conditions de Dirichlet aux deux limites**



Step 1: Temperature differences

$$\begin{cases} e_1 = -\theta_1 \\ e_2 = \theta_1 - \theta_2 \\ e_3 = \theta_2 - \theta_3 \\ e_4 = \theta_3 \end{cases} \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \qquad \mathbf{e} = -\mathbf{A} \cdot \boldsymbol{\theta}$$

Thermal networks: steady-state

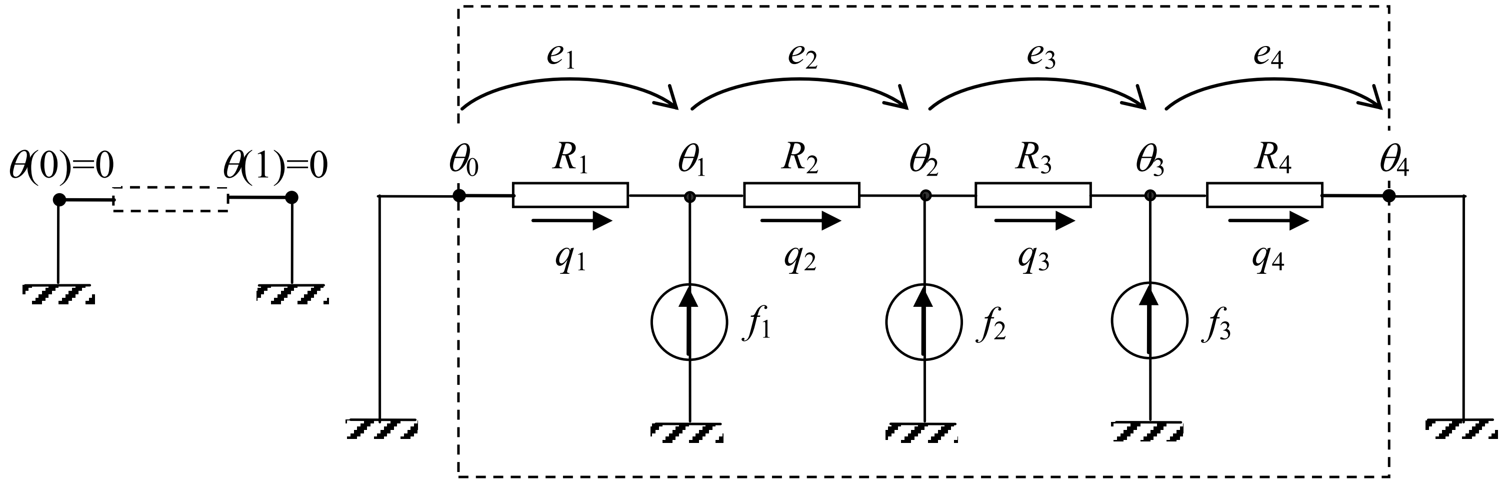


Step 2: Constitutive law (Fourier law)

$$\begin{cases} q_1 = G_1 e_1 \\ q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \\ q_4 = G_4 e_4 \end{cases} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & G_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

$$G_i = 1/R_i,\ i = 1,\ldots,4$$

Thermal networks: steady-state



Step 3: Energy balance

$$\begin{cases} q_1 - q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 - q_4 = -f_3 \end{cases} \qquad -\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \qquad -\mathbf{f} = \mathbf{A}^T \cdot \mathbf{q}$$

Thermal networks: steady-state

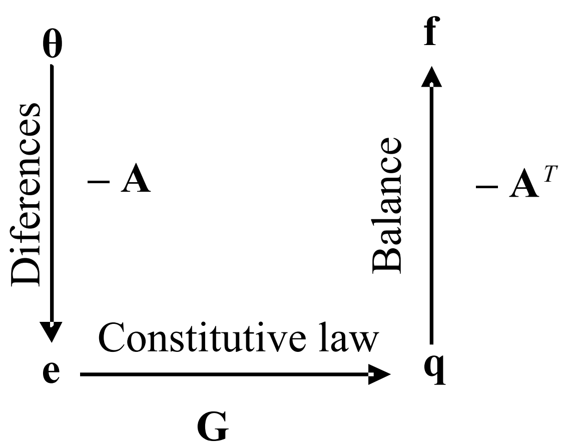
$$\begin{cases} \mathbf{e} = -\mathbf{A}\boldsymbol{\theta} \\ \mathbf{q} = \mathbf{G}\mathbf{e} \\ -\mathbf{A}^T\mathbf{q} = \mathbf{f} \end{cases}$$

$$\begin{cases} \mathbf{G}^{-1}\mathbf{q} + \mathbf{A}\boldsymbol{\theta} = \mathbf{b} \\ -\mathbf{A}^T\mathbf{q} = \mathbf{f} \end{cases}$$

$$\begin{bmatrix} \mathbf{G}^{-1} & \mathbf{A} \\ -\mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{G}\mathbf{A}\boldsymbol{\theta} = \mathbf{f}$$

$$\boldsymbol{\theta} = (\mathbf{A}^T\mathbf{G}\mathbf{A})^{-1}\mathbf{f}$$



- $\boldsymbol{\theta}$ node temperatures
- $\mathbf{e}$ temperature differences over resistances
- $\mathbf{q}$ heat flux through resistances
- $\mathbf{f}$ external heat flow rates

Structural mechanics: dynamics

In **statics** (equilibrium) the masses do not move:
each mass is in balance between external forces f and internal (elastic) forces Kf;
the sum of potential energy is minimum.

In **dynamics**: displacements are changing in time, $u(t)$:
the sum of kinetic energy $\frac{1}{2}mu_t^2$ and energy stored in springs $\frac{1}{2}ce^2$ is constant

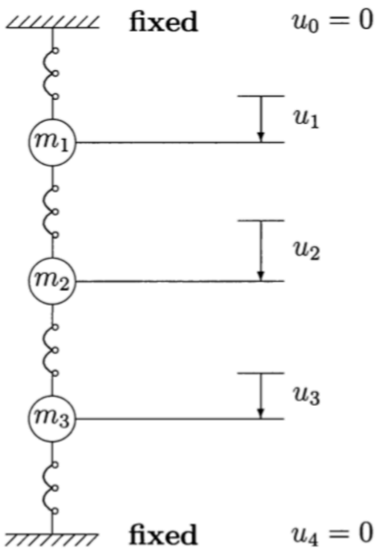
Newton's Law $\boldsymbol{F} = \boldsymbol{ma}$ $\quad f - Ku = Mu_{tt}$ or $\boldsymbol{Mu_{tt}} + \boldsymbol{Ku} = \boldsymbol{f}$

$$F = f - Ku.$$

Displacements $\boldsymbol{u(t)}$
Mass matrix $\boldsymbol{M}$

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{bmatrix}$$

Dynamics



oscillations	$u_1(t), \ldots, u_n(t)$		force balance	$Mu'' + Ku = f(t)$
	$\boldsymbol{A} \downarrow$			$\uparrow \boldsymbol{A}^{\mathrm{T}}$
elongations	$e_1(t), \ldots, e_m(t)$	$\xrightarrow{\boldsymbol{C}}$	spring forces	$w_1(t), \ldots, w_m(t)$

Dynamics

Solution of $\dot{x} = ax$ ($a \in \mathbf{R}$, a constant) is $x(t) = e^{at}x(0)$

Solution of $\dot{\mathbf{x}} = \mathbf{Ax}$ ($\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{A}$ constant) is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$

If $\mathbf{Av} = \lambda\mathbf{v}$, (that is $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$), $\mathbf{v} \neq \mathbf{0}$)
solution of $\dot{\mathbf{x}} = \mathbf{Ax}$ with $\mathbf{x}_0 = \mathbf{v}$, is $\mathbf{x} = e^{\lambda t}\mathbf{v}$

Proof

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0 = e^{\mathbf{A}t}\mathbf{v} =$$

$$\left(\mathbf{I} + t\mathbf{A} + \frac{(t\mathbf{A})^2}{2!} + \cdots\right)\mathbf{v} = \mathbf{v} + \lambda t\mathbf{v} + \frac{(\lambda t)^2}{2!}\mathbf{v} + \cdots = e^{\lambda t}\mathbf{v}$$

Steady-state Dynamics

$$\mathbf{Ku} = \mathbf{f}$$

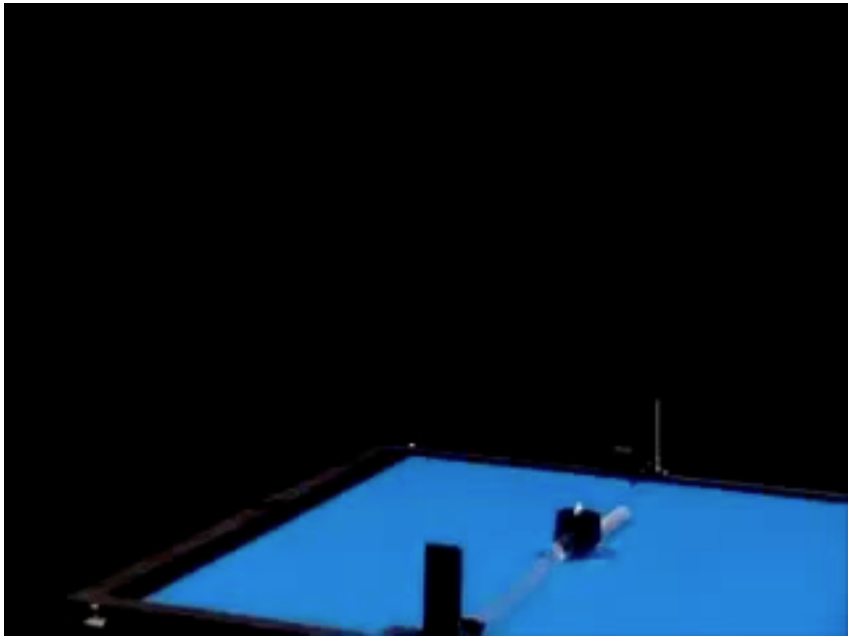
$$\dot{\mathbf{u}} = -\mathbf{Ku} + \mathbf{f}$$

Dynamics

	Oscillator	Diffusion
Differential equation	$\boldsymbol{M}\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x} + \boldsymbol{f}$ $\ddot{x} = -kx$	$\mathbf{C}\dot{\boldsymbol{\theta}} = -\mathbf{K}\boldsymbol{\theta} + \mathbf{f}$ $\dot{\theta} = -k\theta$;
Solution	$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$ $= C_1(\cos\omega t + i\sin\omega t)$ $+ C_2(\cos\omega t - i\sin\omega t)$ $= x_0\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t \; ; \lambda = \omega^2$	$\theta = e^{-kt}\theta_0$

$$\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x} \; ; \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_{\mathbf{2}} \end{bmatrix}$$

$$\mathbf{x} = \left(a_1\cos\sqrt{\lambda_1}t + b_1\sin\sqrt{\lambda_1}t\right)\mathbf{v}_1$$
$$+\left(a_2\cos\sqrt{\lambda_2}t + b_2\sin\sqrt{\lambda_2}t\right)\mathbf{v}_2$$



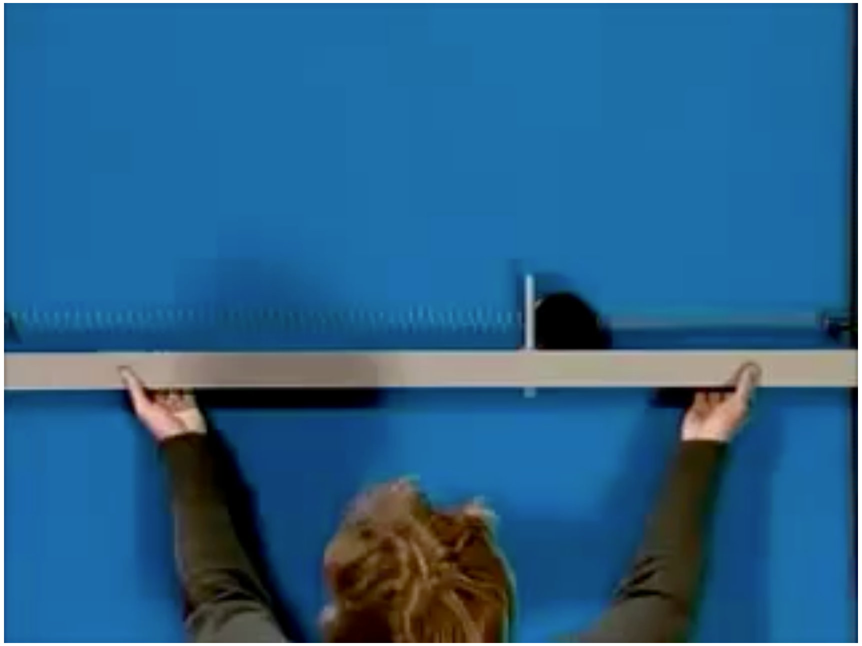
Differential equation:

$\ddot{x} = -\lambda x$;

General solution

$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

$= C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t)$



Particular solution:

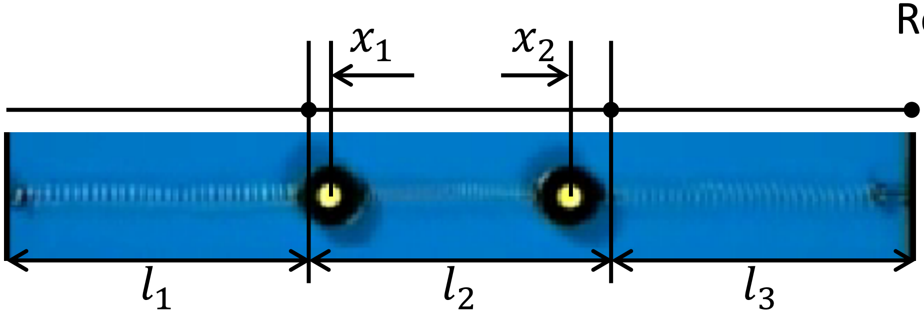
initial condition x_0 and x

$x = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t$; $\lambda = \omega^2$

initial condition x_0 and $x_0 = 0$

$x = x_0 \cos\omega t$

Régime stationnaire



l_i longueur ressorts non-déformés

x_i déplacement des nœuds

Déformation ressorts

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \mathbf{e} = \mathbf{Ax}$$

Forces élastiques dans le ressort

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}; \mathbf{y} = \mathbf{Ce}$$

Equilibre des forces dans chaque nœud

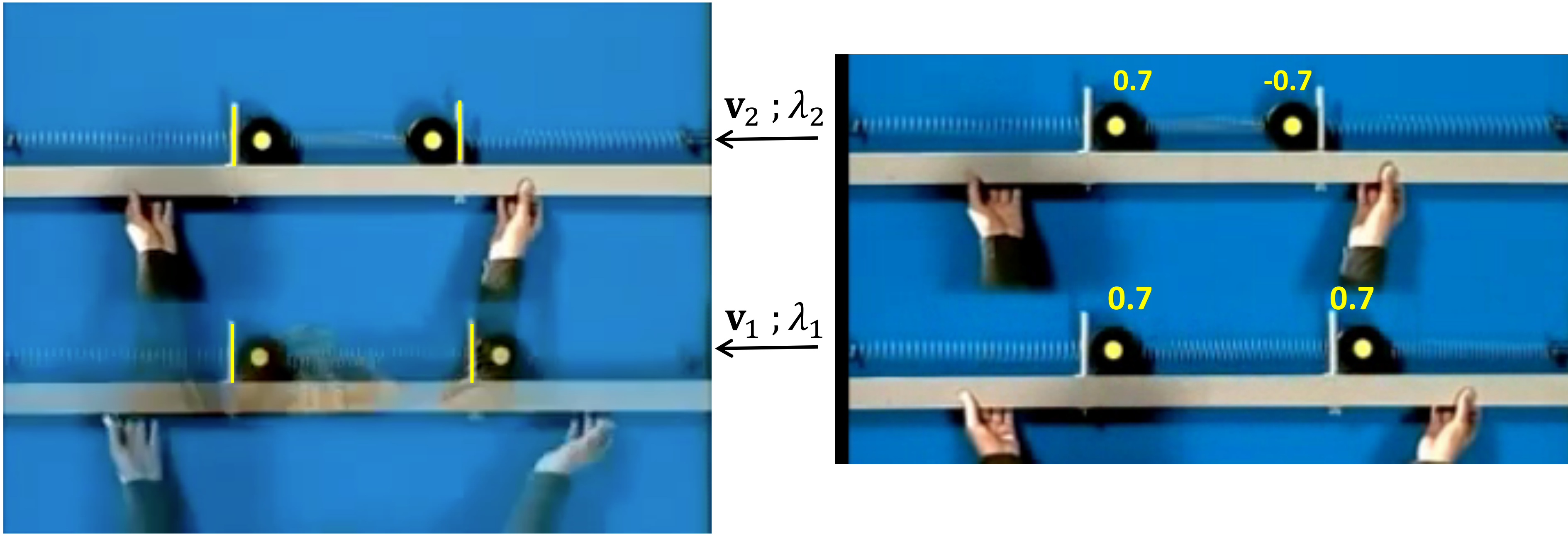
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}; \mathbf{f} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$$

$$\begin{cases} \mathbf{e} = \mathbf{Ax} \\ \mathbf{y} = \mathbf{Ce} \\ \mathbf{f} = \mathbf{A}^{\mathrm{T}}\mathbf{y} \end{cases} \text{ ou}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{CAx} = \mathbf{f}\ ; \mathbf{C} = \mathbf{I}; \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

sans forces externes

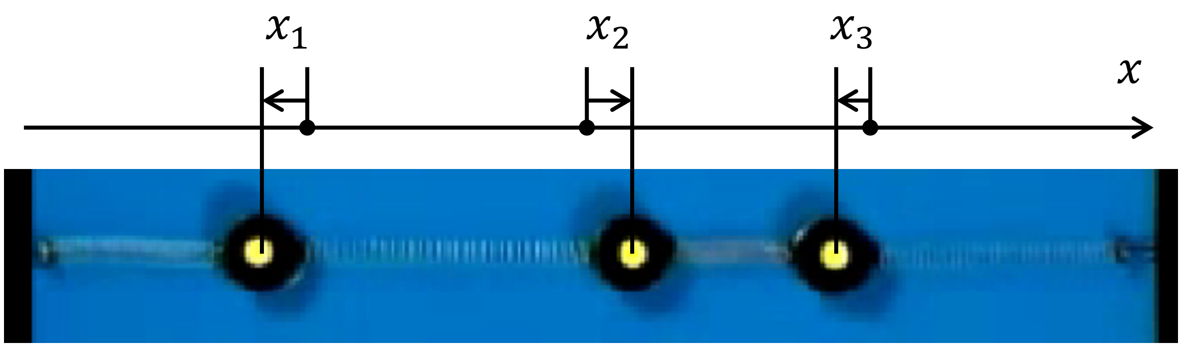
$$\mathbf{A}^{\mathrm{T}}\mathbf{CAx} = \mathbf{0}; \mathbf{Kx} = \mathbf{0}$$



$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} ; \mathbf{V} = \begin{bmatrix} -0.7 & -0.7 \\ -0.7 & 0.7 \end{bmatrix} ; \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\mathbf{x} = a_1 \cos\sqrt{\lambda_1} t\, \mathbf{v}_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \cos t \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\mathbf{x} = a_2 \cos\sqrt{\lambda_2} t\, \mathbf{v}_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_2 \cos\sqrt{3}\, t \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$$



l_i longueur ressorts non-déformés
x_i déplacement des nœuds

Déformation ressorts

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \mathbf{e} = \mathbf{A}\mathbf{x}$$

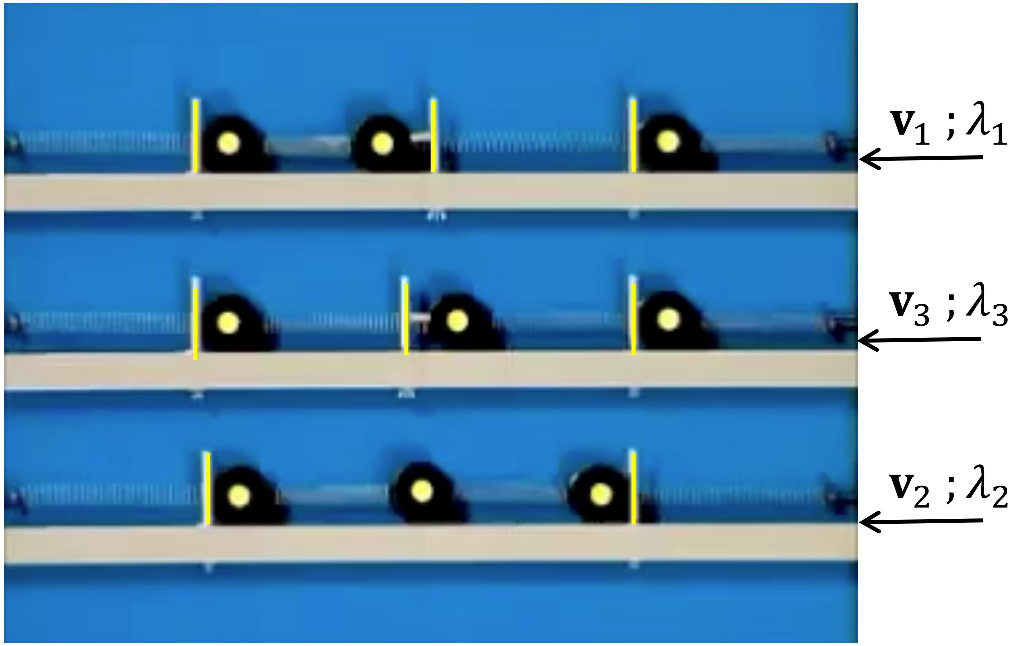
Forces élastiques dans le ressort

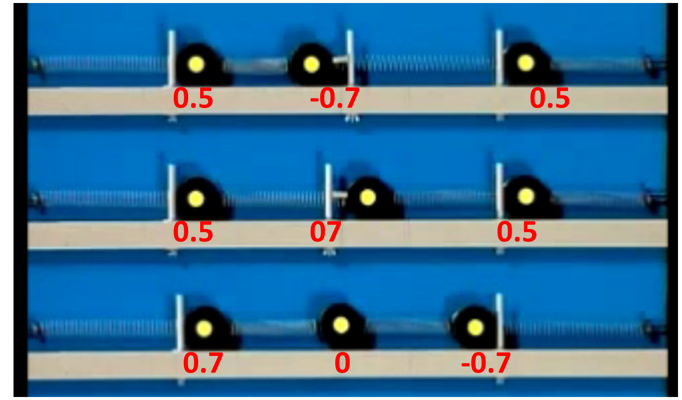
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} ; \mathbf{y} = \mathbf{C}\mathbf{e}$$

Equilibre des forces dans chaque nœud

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} ; \mathbf{f} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$$

$$\begin{cases} \mathbf{e} = \mathbf{A}\mathbf{x} \\ \mathbf{y} = \mathbf{C}\mathbf{e} \\ \mathbf{f} = \mathbf{A}^{\mathrm{T}}\mathbf{y} \end{cases} \quad \text{ou} \quad \mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{A}\mathbf{x} = \mathbf{f}$$





$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}; \mathbf{V} = \begin{bmatrix} 0.5 & -0.7 & -0.5 \\ -0.7 & 0.0 & -0.7 \\ 0.5 & 0.7 & -0.5 \end{bmatrix}; \boldsymbol{\Lambda} = \begin{bmatrix} -3.4 & 0 & 0 \\ 0 & -2.0 & 0 \\ 0 & 0 & -0.6 \end{bmatrix}$$