INSTITUT NATIONAL
DES SCIENCES
APPLIQUÉES
LYON

# Building Energy Simulation

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## Lecture 2: Conduction and Convection

Building Simulation

- Introduction
- Thermal Analysis
- Conduction
  - Constitutive law
  - Heat equation
  - Initial & boundary
  - Thermal resistance
  - Thermal circuits
  - Analysis
    - 1D
    - 2D
    - Dynamic
- Convection
  - Constitutive law
  - Characteristic temp.
  - Heat rate coefficient
- Radiation
- Coupled Transfer

### Curricula

2 x 4h Lectures
- Conduction
- Convection
- Radiation
- Coupled heat transfer

2 x 4h Tutorials and project
- Model your own SmartHome
- Simulate and discuss

1 x 2h Defend your project

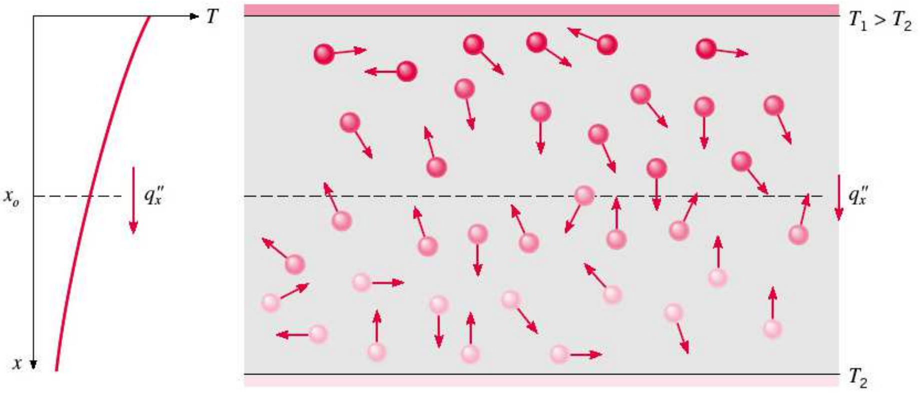
1 x 2h Written exam

### Prerequisites

Calculus
Linear algebra
Thermodynamics
Heat and mass trans

# Conduction
## Constitutive law: Fourier

- **Diffusion**



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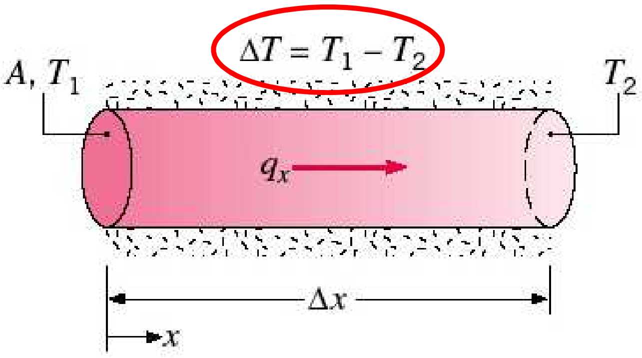
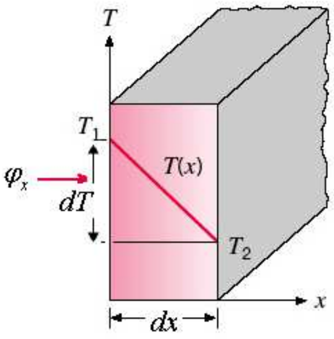
# Conduction
## Constitutive law: Fourier

- **Phenomenological law**
- **Thermal conductivity**
  - property of material
  - measured experimentally

$$\dot{Q}_x \equiv q_x = -\lambda A \frac{dT}{dx}$$

$$\varphi = -\lambda \frac{dT}{dx}$$





# Conduction

## Constitutive law: Fourier

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- **Fourier law:**

$$\boldsymbol{\varphi} = -\lambda \frac{\partial T}{\partial n}\mathbf{n} \qquad \boldsymbol{\varphi} = -\lambda \mathbf{grad}T$$

$$\frac{\partial T}{\partial n}\mathbf{n} \equiv \mathbf{grad}T \equiv \nabla T$$

| Type de système | Coordonnées | Gradient |
|---|---|---|
| Une seule dimension | $\mathbf{n}$ | |
| Trois dimensions | | |
| *Cartésiennes* | $\mathbf{x}, \mathbf{y}, \mathbf{z}$ | $\mathbf{grad}(T) = \frac{\partial T}{\partial x}\mathbf{x} + \frac{\partial T}{\partial y}\mathbf{y} + \frac{\partial T}{\partial z}\mathbf{z}$ |
| *Cylindriques* | $\mathbf{r}, \boldsymbol{\theta}, \mathbf{z}$ | $\mathbf{grad}(T) = \frac{\partial T}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{\theta} + \frac{\partial T}{\partial z}\mathbf{z}$ |
| *Sphériques* | $\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}$ | $\mathbf{grad}(T) = \frac{\partial T}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \omega}\boldsymbol{\omega}$ |

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# Conduction

## Constitutive law: Fourier

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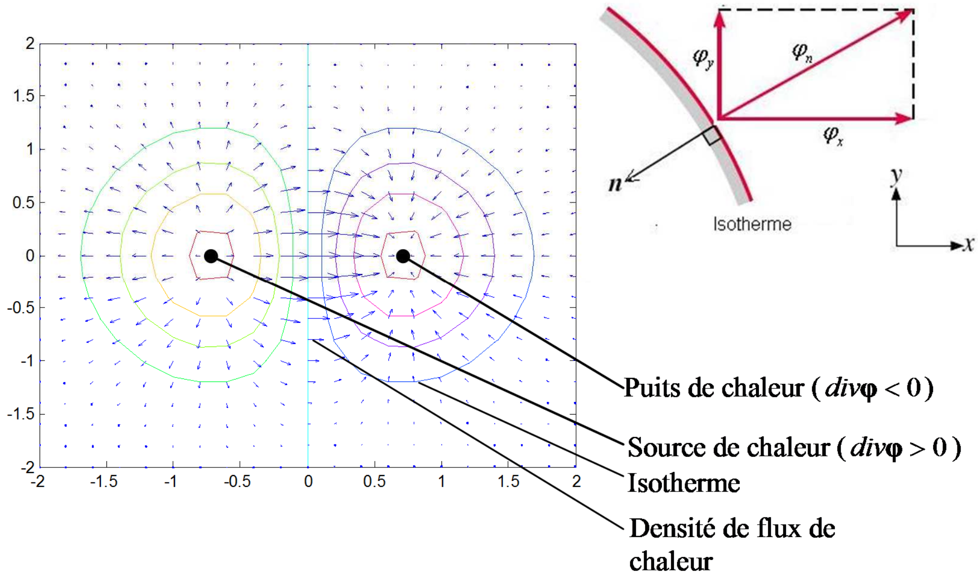
- **temperature distribution: scalar**
- **heat field: vector**



# Conduction
## Heat equation

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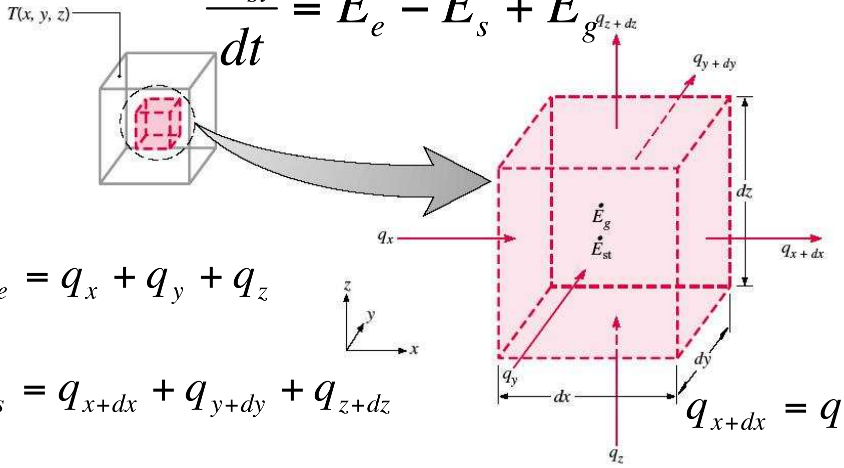
- **Energy balance for a control volume**

$$\frac{E_{st}}{dt}=\dot{E}_e-\dot{E}_s+\dot{E}_g$$



$$\dot{E}_e=q_x+q_y+q_z$$

$$\dot{E}_s=q_{x+dx}+q_{y+dy}+q_{z+dz}$$

$$q_{x+dx}=q_x+\frac{\partial q_x}{\partial x}dx$$

$$\underbrace{-\frac{\partial q_x}{\partial x}dx-\frac{\partial q_y}{\partial y}dy-\frac{\partial q_z}{\partial z}dz}_{\dot{E}_e-\dot{E}_s}+\underbrace{p\,dx\,dy\,dz}_{\dot{E}_g}=\underbrace{\rho c\,dx\,dy\,dz\frac{\partial T}{\partial t}}_{\dot{E}_{st}}$$

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# Conduction
## Heat equation

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- **Energy balance: differential equation**

$$\varphi_x\equiv\frac{q_x}{dy\,dz}\qquad\varphi_y\equiv\frac{q_y}{dx\,dz}\qquad\varphi_z\equiv\frac{q_z}{dx\,dy}$$

$$\underbrace{-\frac{\partial q_x}{\partial x}dx-\frac{\partial q_y}{\partial y}dy-\frac{\partial q_z}{\partial z}dz}_{\dot{E}_e-\dot{E}_s}+\underbrace{p\,dx\,dy\,dz}_{\dot{E}_g}=\underbrace{\rho c\,dx\,dy\,dz\frac{\partial T}{\partial t}}_{\dot{E}_{st}}$$

$$-div\boldsymbol{\varphi}+p=\rho c\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t}+div\left(\frac{\boldsymbol{\varphi}}{\rho c}\right)=\frac{p}{\rho c}$$

# Conduction
## Heat equation

- **Heat equation**

$$\frac{\partial T}{\partial t} + div\left(\frac{\boldsymbol{\varphi}}{\rho c}\right) = \frac{p}{\rho c}$$

continuity equation (fundamental)

$$\boldsymbol{\varphi} = -\lambda \mathbf{grad} T$$

Fourier law (empirical)

$$\rho c \frac{\partial T}{\partial t} = div(\lambda\, \mathbf{grad} T) + p$$

heat equation

$$\frac{\partial T}{\partial t} = div(\alpha\, \mathbf{grad} T) + \frac{p}{\rho c}$$

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# Conduction
## Heat equation

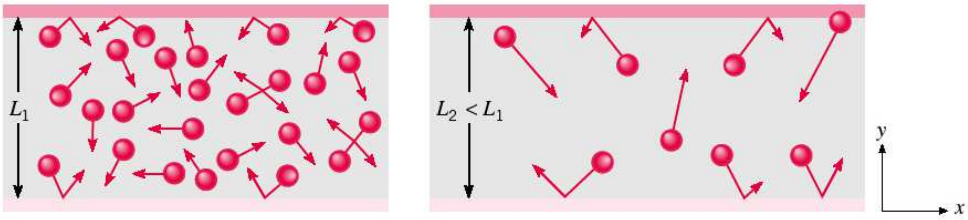
- **Note on empiricism : micro-scale effects**

$$\boldsymbol{\varphi} = -\lambda\, \mathbf{grad} T$$

Fourier law is empirical:

- boundary effects at micro- and nano-scales

- nano-structured materials



# Conduction
## Heat equation

- **Schéma du raisonnement**

$$\begin{tikzcd}
T \arrow[d, "\mathbf{grad}", "\text{Differences}"'] & & div\boldsymbol{\varphi} = p - \rho c \dfrac{\partial T}{\partial t} \\
\mathbf{grad}T \arrow[rr, "\text{Constitutive law}", "-\lambda"'] & & \boldsymbol{\varphi} = -\lambda \mathbf{grad}T \arrow[u, "\text{Balance}", "div"']
\end{tikzcd}$$

| Symbol | Significance |
|---|---|
| $T$ | Potential |
| $\mathbf{grad}\,T$ | Difference of potential |
| $\boldsymbol{\varphi} = -\lambda\,\mathbf{grad}\,T$ | Heat flow rate due to potential difference |
| $p$ | External heat flow rates |

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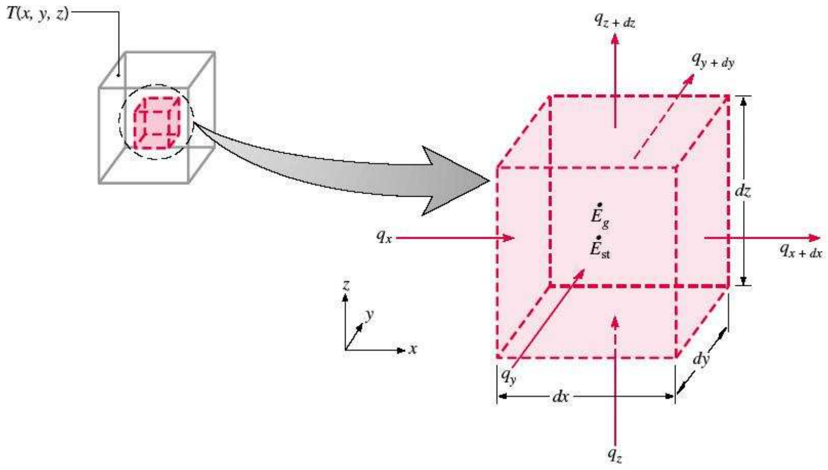
# Conduction
## Heat equation

- **Cartesian coordinate system**

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+p=\rho\,c\frac{\partial T}{\partial t}$$



# Conduction
## Heat equation

- **Homogeneous and isotropic material** $\lambda = const.$

$$\rho c \frac{\partial T}{\partial t} = \lambda \, div(\mathbf{grad}T) + p$$

$$div(\mathbf{grad}T) = \Delta T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$ Laplacien

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = div(\mathbf{grad}T) + \frac{p}{\lambda}$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \Delta T + \frac{p}{\lambda}$$

$$\alpha = \frac{\lambda}{\rho c}$$ Diffusivity

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# Conduction
## Heat equation

- **Homogeneous and isotropic material** $\lambda = const.$

**in steady-state** $\frac{\partial T}{\partial t} = 0$

**Poisson equation:**

$$\lambda \, div(\mathbf{grad}T) + p = 0$$

$$\lambda \, \Delta T + p = 0$$

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## Heat equation

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- **Homogeneous and isotropic material** $\lambda = const.$

  **in steady-state** $\frac{\partial T}{\partial t} = 0$

  **without internal sources** $p = 0$

  **Laplace equation:**

$$div(\mathbf{grad}T) = 0$$

$$\Delta T = 0$$

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# Conduction

## Initial and boundary conditions

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$$\rho c \frac{\partial T}{\partial t} = div(\lambda \mathbf{grad}T) + p$$

For each dimension:

2 conditions at the boundaries

Space distribution of temperature
at initial time

# Conduction

## Initial and boundary conditions

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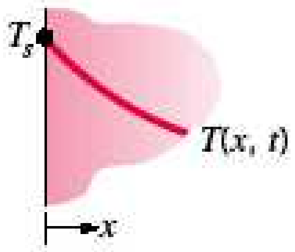
- **Initial conditions**

$$T\Big|_{t=0} \equiv T_0 = f(x, y, z, 0)$$

- **Dirichelt boundary conditions**

$$T(0,t) = T_S$$



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## Initial and boundary conditions

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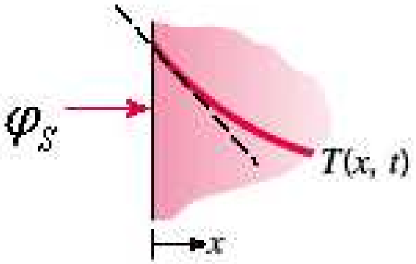
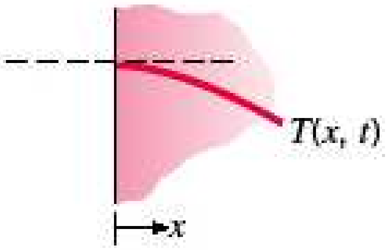
- **Neumann boundary condition**
  - imposed heat flux

$$-\lambda \frac{\partial T}{\partial x}\Big|_{x=0} = \varphi_S$$



  - adiabatic surface or symmetry

$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$



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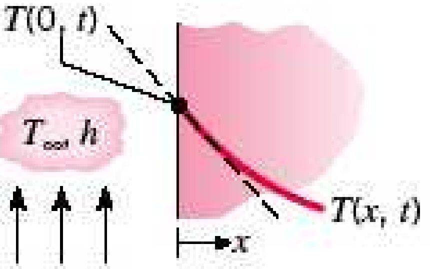
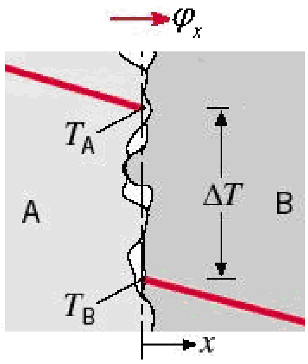
- **Fourier boundary condition**

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = h_c[T_\infty - T(0,t)]$$



- **Surfaces in contact**

$$\lambda_A \left.\frac{\partial T}{\partial x}\right|_{x=0^-} = \lambda_B \left.\frac{\partial T}{\partial x}\right|_{x=0^+}$$



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## Thermal resistance

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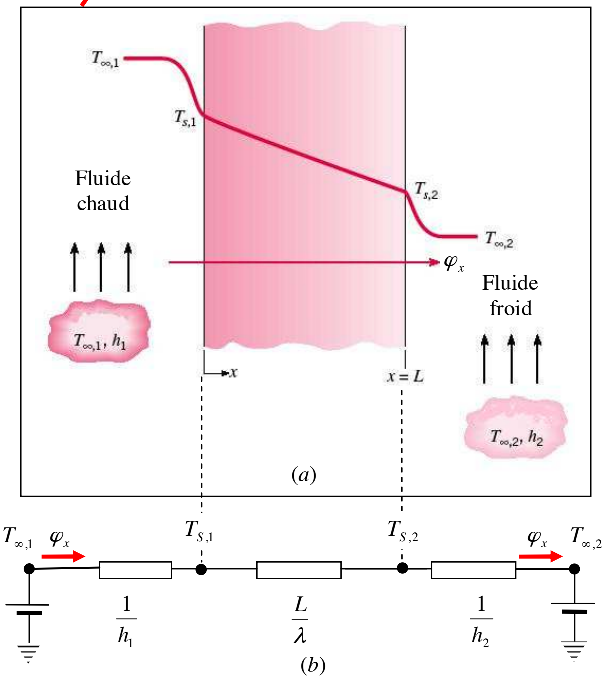
$$\cancel{\rho c \frac{\partial T}{\partial t}} = div(\lambda \mathbf{grad} T) + \cancel{p}$$

$$div(\lambda\ \mathbf{grad} T) = 0$$

- **Plane wall**

$$\begin{cases} \dfrac{d}{dx}\left(\lambda \dfrac{dT}{dx}\right) = 0 \\ T(0) = T_{s1} \\ T(L) = T_{s2} \end{cases}$$

$$T(x) = C_1 x + C_2$$



# Conduction

## Thermal resistance

- **Boundary conditions**

$$T(0) = T_{S1} \qquad T(L) = T_{S2}$$

➔ constants of integration

$$C_2 = T_{S1} \qquad C_1 = \frac{T_{S2} - T_{S1}}{L}$$

➔ particular solution

$$T(x) = \frac{T_{S2} - T_{S1}}{L} x + T_{S1}$$

- **Fourier law ➔**

$$\varphi_x = -\frac{\lambda}{L}(T_{S2} - T_{S1}) = \frac{\lambda}{L}(T_{S1} - T_{S2})$$

- **Heat flux**

$$q_x = A\varphi = \lambda \frac{A}{L}(T_{S1} - T_{S2})$$

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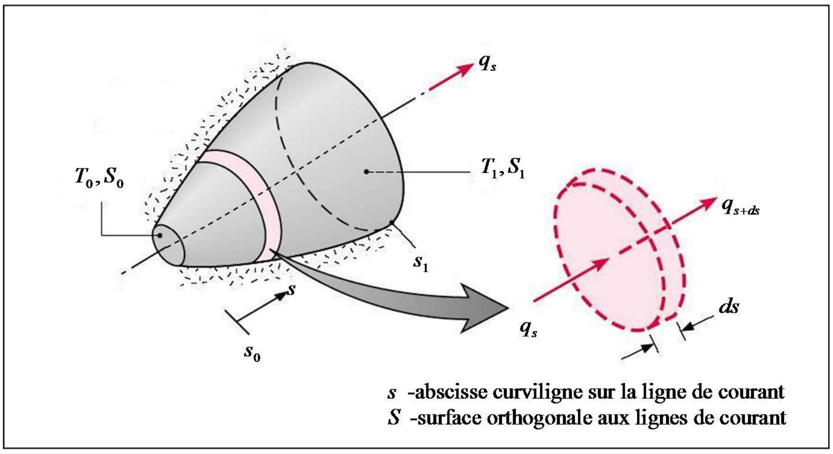
# Conduction

## Thermal resistance



$s$ -abscisse curviligne sur la ligne de courant
$S$ -surface orthogonale aux lignes de courant

*Steady-state conduction without internal sources*

- **Fourier law in a section**

$$q = S(s)\,\varphi = -S(s)\,\lambda(s)\left.\frac{dT}{ds}\right|_s$$

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- Separation of variables

$$q\frac{ds}{S(s)\,\lambda(s)} = -dT$$

$$q\int_{s_0}^{s}\frac{ds}{S(s)\lambda(s)} = -\int_{T_0}^{T} dT \quad \Leftrightarrow \quad qR = T_0 - T_1$$

$$R = \int_{s_0}^{s}\frac{ds}{S(s)\lambda(s)} \qquad R_{cd} \equiv \frac{T_0 - T_1}{q}$$

$$R_{cd} = \frac{1}{\lambda}\frac{L}{A} \qquad R_{cd} = \frac{1}{\lambda}\frac{\ln(r_2/r_1)}{2\pi L} \qquad R_{cd} = \frac{1}{\lambda}\frac{1}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Plane wall  cylinder  sphere

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# Conduction
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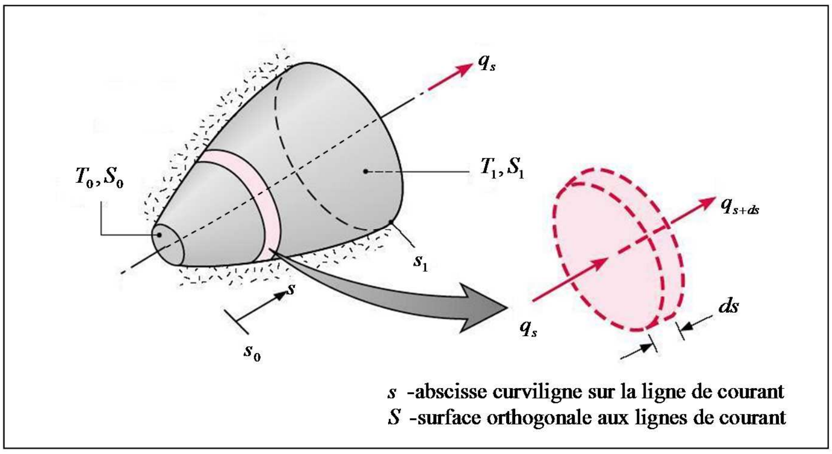
- **Stream surface (tube)**



$s$ -abscisse curviligne sur la ligne de courant
$S$ -surface orthogonale aux lignes de courant

$$q_s - q_{s+ds} + p\,S\,ds = 0$$ Steady-state conduction

$$dq = p\,S\,ds$$

$$\int_{q_0}^{q} dq = \int_{s_0}^{s} p\,S\,ds \qquad \boxed{q(s) = \int_{s_0}^{s} pS\,ds + q_0}$$

# Conduction
## Thermal circuits

- **stream surface (tube)**

$$q(s) = -\lambda S \frac{dT}{ds}$$

$$q(s) = \int_{s_0}^{s} pS\, ds + q_0$$

$$\lambda S \frac{dT}{ds} = -\int_{s_0}^{s} p\, S\, ds - q_0$$

$$dT = \frac{1}{\lambda S}\left[-\int_{s_0}^{s} p\, S\, ds'\right] ds - q_0 \frac{ds}{\lambda S}$$

$$T_1 = \int_{s_0}^{s_1} \frac{1}{\lambda S}\left[\int_{s_0}^{s} - p\, S\, ds'\right] ds - q_0 \int_{s_0}^{s_1} \frac{ds}{\lambda S} + T_0$$

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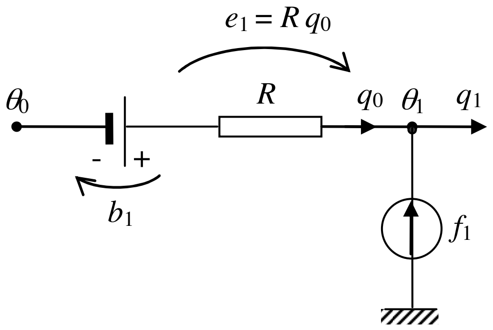
# Conduction
## Thermal circuits

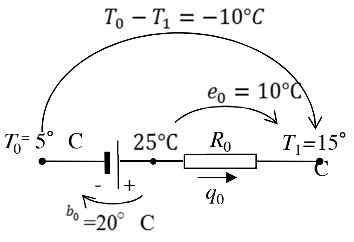
$$dr = \frac{ds}{\lambda S} \qquad R = \int_{s_0}^{s_1} \frac{ds}{\lambda S}$$

$$T_1 = -r_1 \int_{s_0}^{s_1} p\, S\, ds + \int_{s_0}^{s_1} p\, S\, r\, ds - R\, q_0 + T_0$$

$$f_1 = \int_{s_0}^{s_1} pS\, ds \qquad b_1 = -r_1 \int_{s_0}^{s_1} pS\, ds + \int_{s_0}^{s_1} pSr\, ds$$

$$\begin{cases} q_1 = q_0 + f_1 \\ \theta_1 = b_1 + e_1 + \theta_0 \end{cases}$$





$$e_0 = T_0 + b_0 - T_1$$

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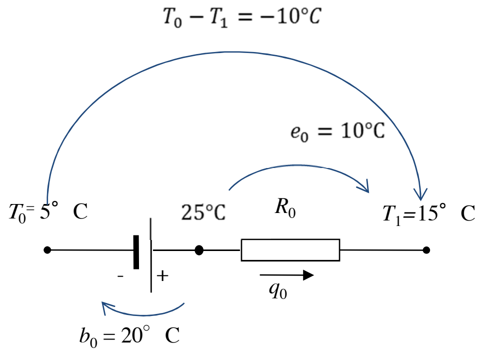

$$e_0 = T_0 + b_0 - T_1$$

- **Convection resistance**

$$R_{cv} \equiv \frac{T_S - T_\infty}{q} = \frac{1}{h_{cv} A}$$

- **Radiative resistance**

$$R_r \equiv \frac{T_S - T_{me}}{q} = \frac{1}{h_r A}$$

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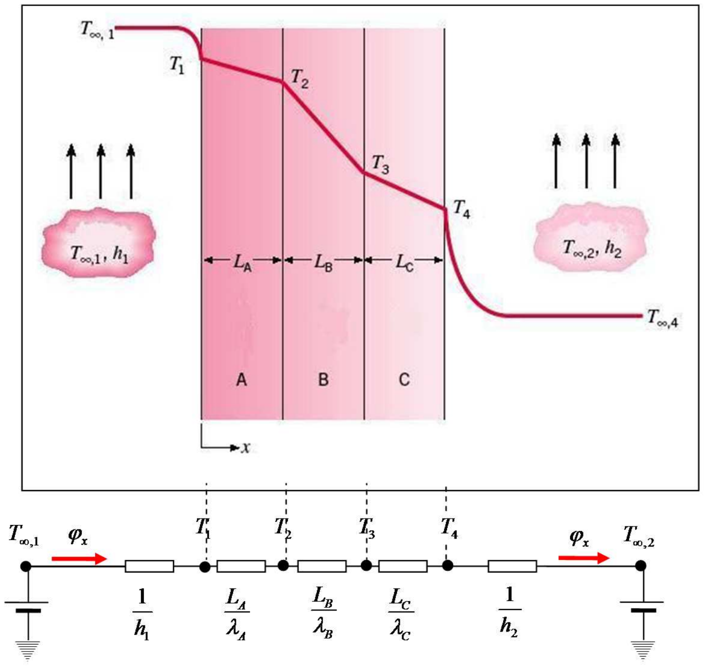
- **Multilayer 1-D wall**



# Conduction
## Thermal circuits

- **Mur multicouche**

$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{\lambda_A A} + \frac{L_B}{\lambda_B A} + \frac{L_C}{\lambda_C A} + \frac{1}{h_2 A}$$

$$q_x \equiv U \cdot A \cdot \Delta T$$

$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{\lambda_A} + \frac{L_B}{\lambda_B} + \frac{L_C}{\lambda_C} + \frac{1}{h_2}}$$

$$q_x = \frac{T_{\infty 1} - T_{\infty 2}}{R_{tot}} = UA(T_{\infty 1} - T_{\infty 2})$$

$$\varphi_x = U(T_{\infty 1} - T_{\infty 2}) = \frac{T_{\infty 1} - T_{\infty 2}}{\frac{1}{h_1} + \frac{L_A}{\lambda_A} + \frac{L_B}{\lambda_B} + \frac{L_C}{\lambda_C} + \frac{1}{h_2}}$$

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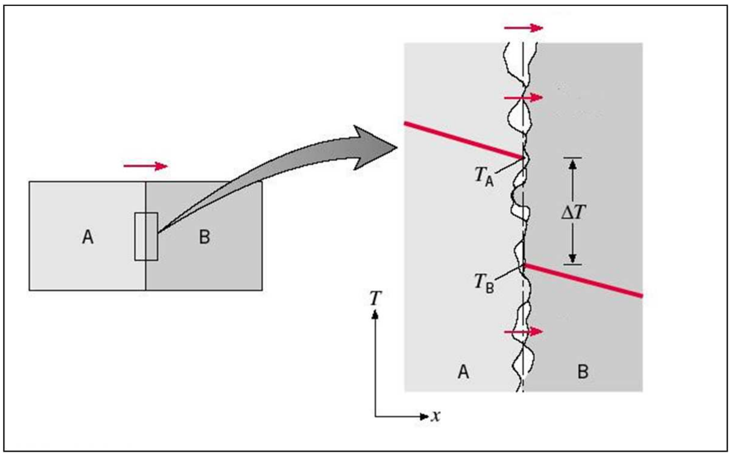
# Conduction
## Thermal circuits

- **Contact resistance**



$$R_{ct} \equiv \frac{T_A - T_B}{q_x}$$

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## Thermal circuits

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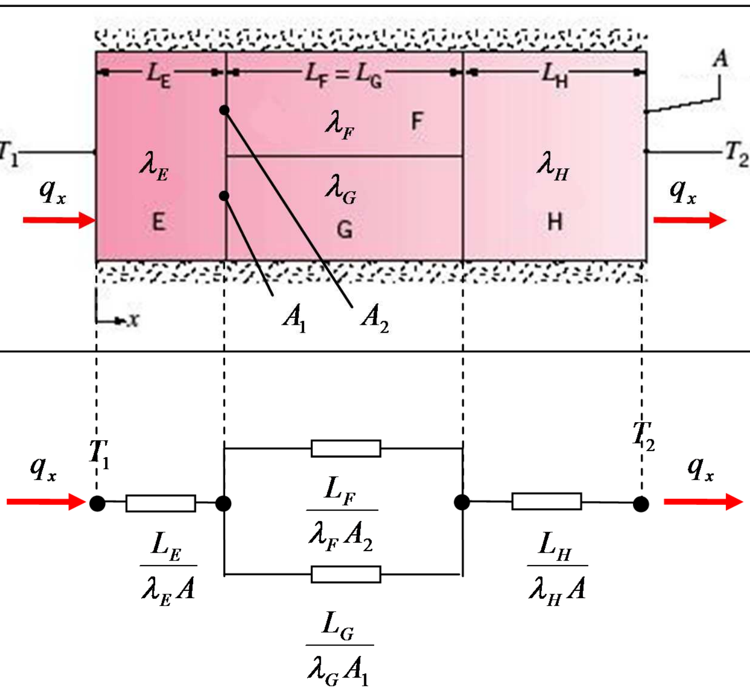
- **Example circuit**



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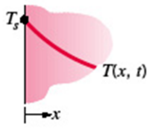
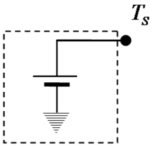
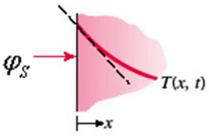
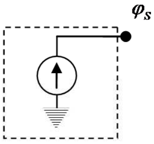
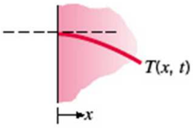
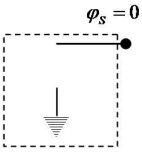
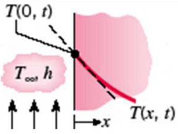
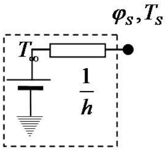
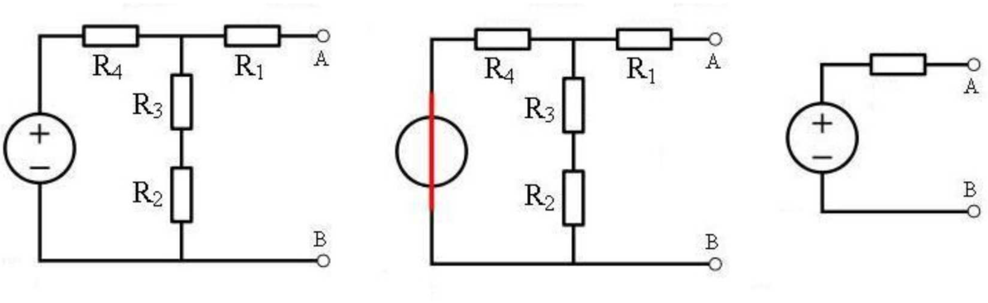
| | Type de condition | Equation | Variation de la température | Source équivalente |
|---|---|---|---|---|
| 1. | Température imposée sur la surface (condition de Dirichlet) | $T(0,t) = T_S$ |  |  |
| 2. | Densité de flux imposée sur la surface (condition de Neumann) | | | |
| | a. Densité de flux imposée | $-\lambda \frac{\partial T}{\partial x}\Big\vert_{x=0} = \varphi_S$ |  |  |
| | b. Surface adiabatique ou surface de symétrie | $\frac{\partial T}{\partial x}\Big\vert_{x=0} = \varphi_S = 0$ |  |  |
| 3. | Surface avec convection (condition de Fourier) | $-\lambda \frac{\partial T}{\partial x}\Big\vert_{x=0} \equiv \varphi_S$<br>$= h[T_\infty - T(0,t)]$<br>$= h(T_\infty - T_S)$ |  |  |

# Conduction
## Thermal circuits

- **Thévenin theorem**
  - tension ← potential diff.
  - resistance ← passivized sources



$$q_{tot} = \frac{T}{R_1 + \left( \frac{1}{R_2} + \frac{1}{R_3} \right)^{-1}}$$

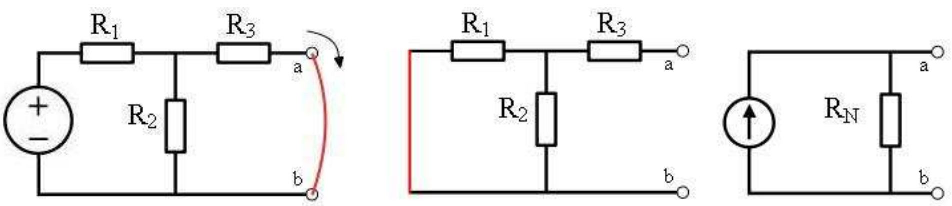
$$T_T = \frac{R_2 + R_3}{(R_2 + R_3) + R_4} T_1$$

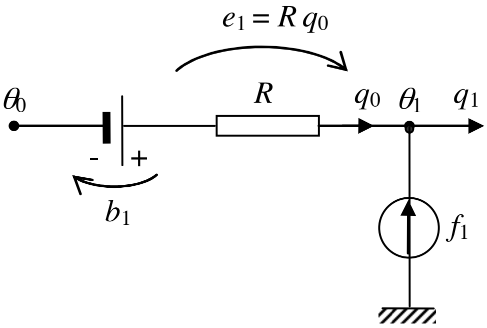
$$R_T = R_1 + \left( \frac{1}{R_2 + R_3} + \frac{1}{R_4} \right)^{-1}$$

---

# Conduction
## Thermal circuits

- **Norton theorem**
  - current ← short circuit terminals
  - resistance ← passivized sources





$$q_N = \frac{R_2}{R_2 + R_3} q_{tot}$$

$$R_N = R_3 + \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}$$

# Conduction
## Analysis of thermal circuits 1D

- **Heat equation**

$$\rho c \frac{\partial T}{\partial t} = \lambda \, div(\mathbf{grad} T) + p$$

- **Homogenous and isotropic material**

$$\lambda = const.$$

**steady-state**

$$\frac{\partial T}{\partial t} = 0$$

**Poisson equation:**

$$\lambda \, div(\mathbf{grad} T) + p = 0 \qquad \lambda \, \Delta T + p = 0$$

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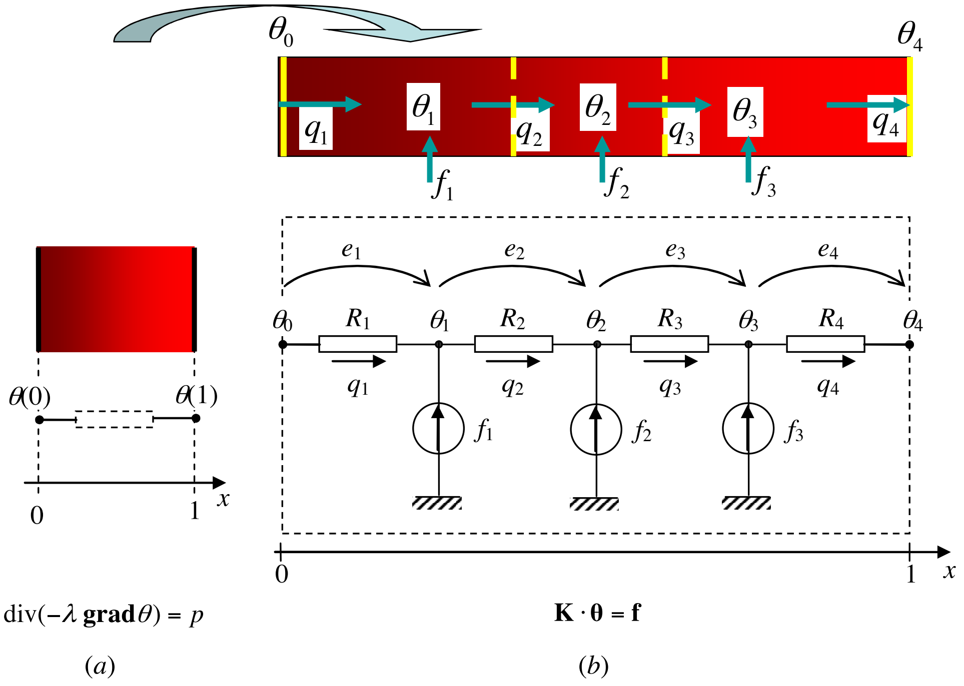
# Conduction
## Analysis of thermal circuits 1D

- **Poisson equation**

$$\text{div}(-\lambda \, \mathbf{grad}\, \theta) = p$$



# Conduction

## Analysis of thermal circuits 1D

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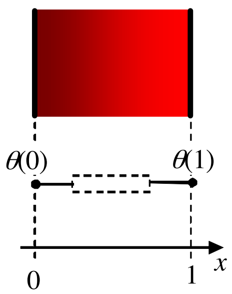
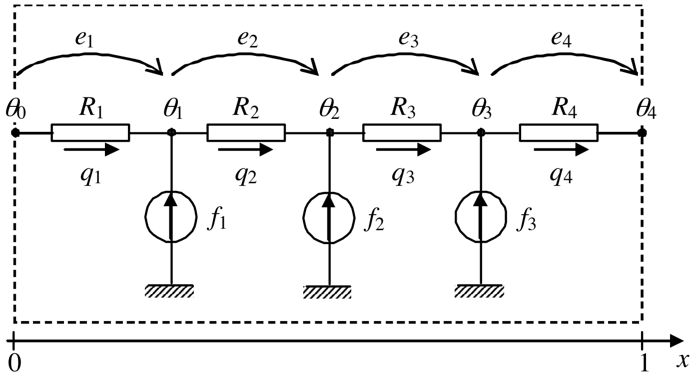
- **Direct problem**



$$\mathrm{div}(\lambda\,\mathbf{grad}\,\theta) = -p$$

(*a*)



$$\mathbf{A}^T\mathbf{C}\mathbf{A}\cdot\boldsymbol{\theta} = -\mathbf{f}$$

(*b*)

- **given**: circuit and boundary conditions
- **find**: node temperatures and heat flow rates on branches

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# Conduction

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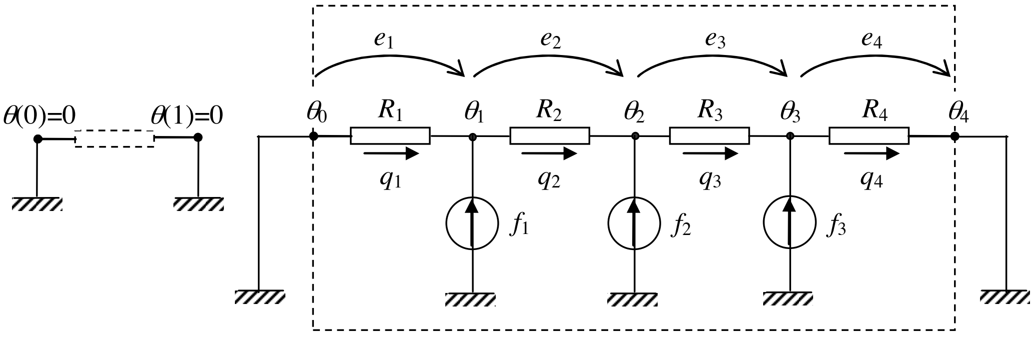
- **Dirichlet boundary conditions**

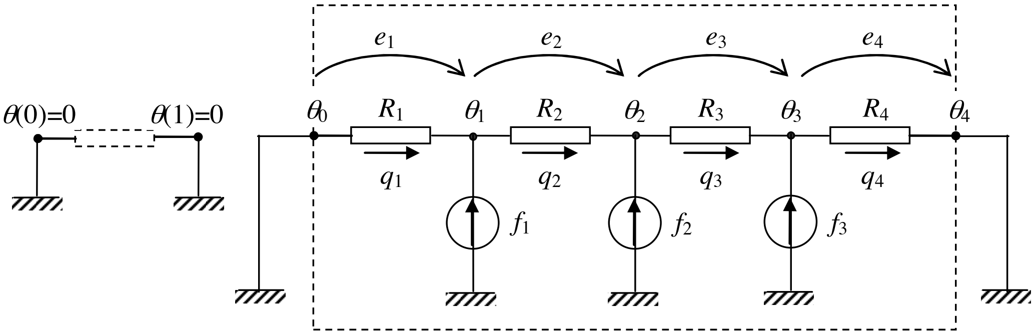


  – temperature differences for each resistance

$$\begin{cases} e_1 = -\theta_1 \\ e_2 = \theta_1 - \theta_2 \\ e_3 = \theta_2 - \theta_3 \\ e_4 = \theta_3 \end{cases} \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \qquad \mathbf{e} = -\mathbf{A}\cdot\boldsymbol{\theta}$$

# Conduction
## Analysis of thermal circuits 1D

- **Dirichlet boundary conditions**



  – heat flux : constitutive law (Fourier)

$$\begin{cases} q_1 = G_1 e_1 \\ q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \\ q_4 = G_4 e_4 \end{cases} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & G_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

$$G_i = 1/R_i, i = 1,\ldots,4$$

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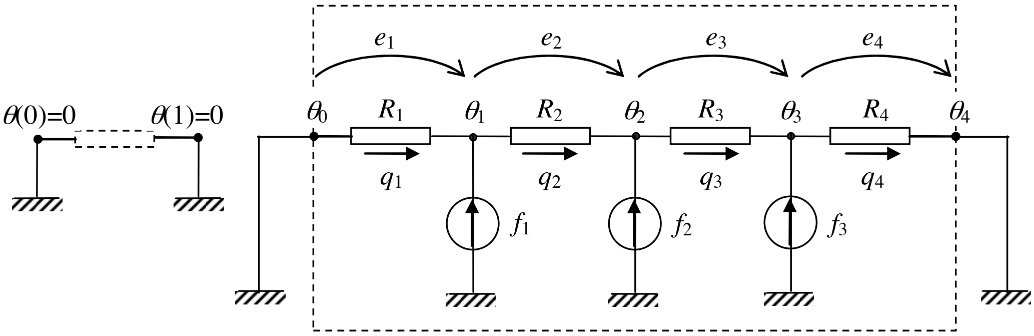
# Conduction
## Analysis of thermal circuits 1D

- **Dirichlet boundary conditions**



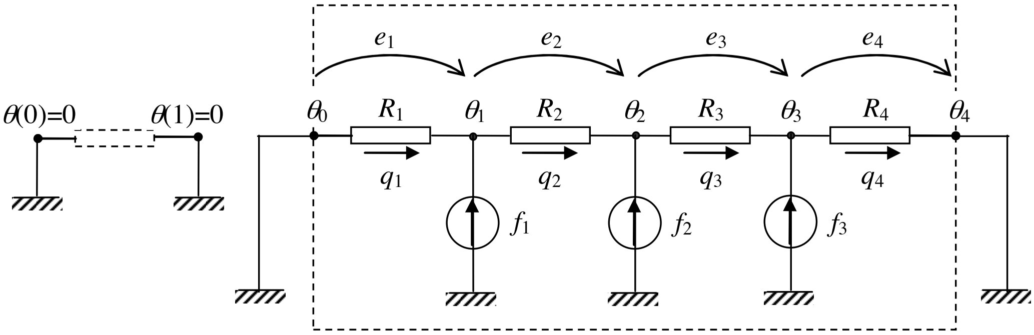
  – heat balance in nodes

$$\begin{cases} q_1 - q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 - q_4 = -f_3 \end{cases} \qquad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_{'} \end{bmatrix} \qquad \mathbf{f} = -\mathbf{A}^T \cdot \mathbf{q}$$

# Conduction

## Analysis of thermal circuits 1D

- **Dirichlet boundary conditions**



  – solution

$$\mathbf{K} \cdot \boldsymbol{\theta} = \mathbf{f} \qquad \mathbf{K} = \mathbf{A}^T\mathbf{G}\mathbf{A} = \begin{bmatrix} G_1 + G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 & -G_3 \\ 0 & -G_3 & G_3 + G_4 \end{bmatrix}$$

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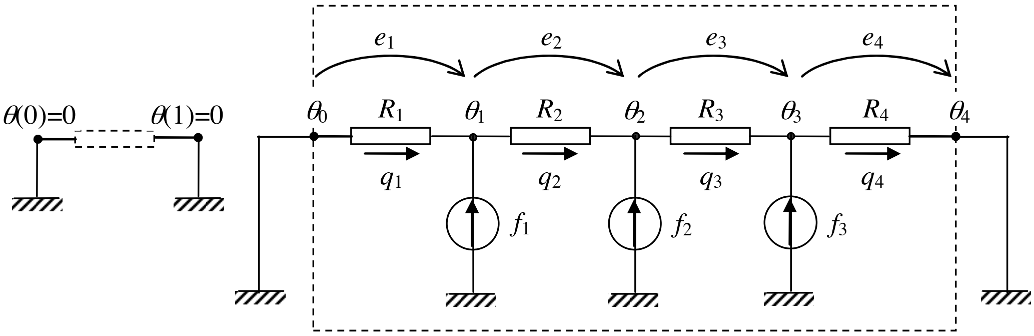
# Conduction

## Analysis of thermal circuits 1D

- **Dirichlet boundary conditions**



$$\mathbf{f} = \mathbf{K} \cdot \boldsymbol{\theta} \xrightarrow{\qquad} \boldsymbol{\theta} = \mathbf{K}^{-1} \cdot \mathbf{f}$$

$$\mathbf{G} = \mathbf{I}$$

$$\mathbf{K} \equiv \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

# Conduction
## Analysis of thermal circuits 1D

- **Dirichlet boundary conditions**

$$\mathbf{K} \equiv \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- **Properties of K matrix**
  - symmetric $\mathbf{K} = \mathbf{K}^T$
  - tri-diagonal
  - sparse
  - constant diagonal

- **Toeplitz matrix**

---

# Conduction
## Analysis of thermal circuits

- **Framework for analysis of thermal circuits**

$$\begin{tikzcd}
\boldsymbol{\theta} \arrow[d, "-\mathbf{A}", "\text{Connexions}"'] & -\mathbf{A}^T\mathbf{q} = \mathbf{f} \\
\mathbf{e} = -\mathbf{A}\boldsymbol{\theta} \arrow[r, "\text{Loi constitutive}", "\mathbf{G}"'] & \mathbf{q} = \mathbf{G}\mathbf{e} \arrow[u, "\text{Bilans}", "-\mathbf{A}^T"']
\end{tikzcd}$$

**θ** températures dans les nœuds
**e** chute de température sur les résistances
**q** flux dans les résistances
**f** flux externes

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- **Equivalence between continuous and discrete**

$$\begin{tikzcd} T \arrow[d, "\mathbf{grad}", "\text{Connexion}"'] & f = p - \rho c \frac{\partial T}{\partial t} \\ \nabla T \arrow[r, "\text{Loi constitutive}", "-\lambda"'] & \boldsymbol{\varphi} \arrow[u, "\text{Bilan}", "div"'] \end{tikzcd}$$

$$\begin{tikzcd} \boldsymbol{\theta} \arrow[d, "-\mathbf{A}", "\text{Connexions}"'] & \mathbf{A}^T\mathbf{q} = \mathbf{f} \\ \mathbf{e} = -\mathbf{A}\boldsymbol{\theta} \arrow[r, "\text{Loi constitutive}", "\mathbf{G}"'] & \mathbf{q} = \mathbf{Ge} \arrow[u, "\text{Bilans}", "-\mathbf{A}^T"'] \end{tikzcd}$$

| | Continu | Discret |
|---|---|---|
| Equation de la chaleur | $\mathrm{div}(\lambda\,\mathbf{grad}\,\theta) = -p$ | $\mathbf{A}^T\mathbf{G}\mathbf{A}\ \boldsymbol{\theta} = \mathbf{f}$ |
| Conditions aux limites | en $x = 0 : \theta(0)$ ou $\varphi(0)$<br>en $x = 1 : \theta(1)$ ou $\varphi(1)$ | incluses dans<br>la matrice $\mathbf{A}^T\mathbf{G}\mathbf{A}$ |
| Température | $\theta$ | $\boldsymbol{\theta}$ |
| Flux | $\varphi$ | $\mathbf{q}$ |
| Connexion | $-\,\mathbf{grad}\,\theta$ | $-\,\mathbf{A}\boldsymbol{\theta}$ |
| Bilan d'énergie | -div $\varphi$ | $-\,\mathbf{A}^T\mathbf{q}$ |
| Constante de matériau | $\lambda$ | $\mathbf{G}$ |
| Sources | $p$ | $-\,\mathbf{f}$ |

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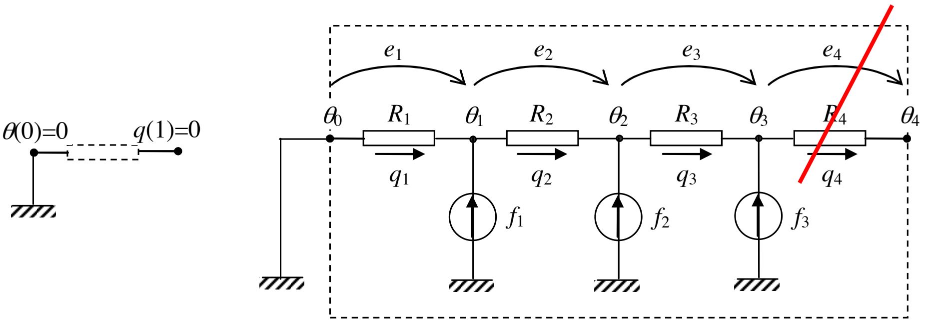
- **Dirichlet plus Neumann**



  - Connectivity matrix: circuit topology

$$\begin{array}{r|ccc|l} \text{nœud} & \theta_1 & \theta_2 & \theta_3 & \text{arc} \\ \mathbf{A} = & 1 & 0 & 0 & q_1 \\ & -1 & 1 & 0 & q_2 \\ & 0 & -1 & 1 & q_3 \\ & & & & \cancel{q_4} \end{array}$$

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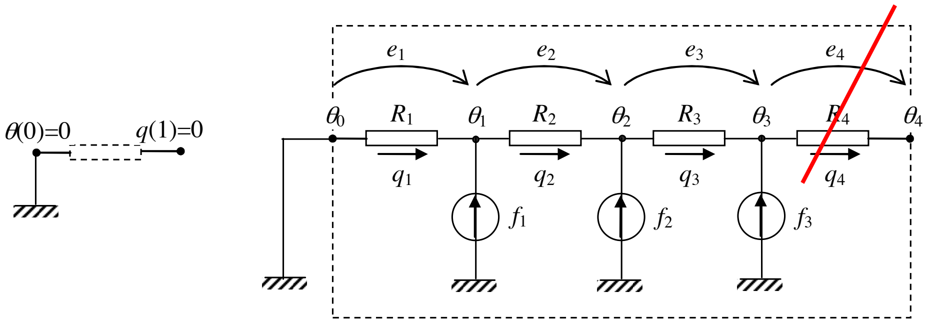
- **Dirichlet plus Neumann**



  - flux: constitutive law

$$\begin{cases} q_1 = G_1 e_1 \\ q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \end{cases} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

$$q_4 = G_4 e_4 = 0 \qquad \text{(connu)}$$

$$G_i = 1 / R_i, i = 1,\ldots,4$$

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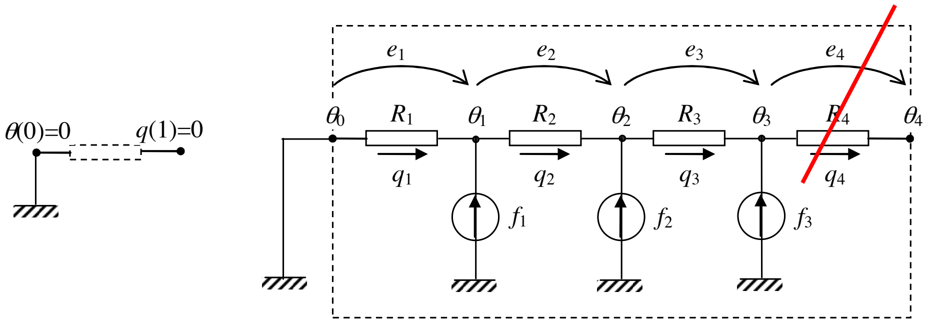
- **Dirichlet plus Neumann**



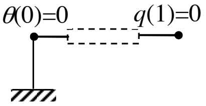
  - heat balance

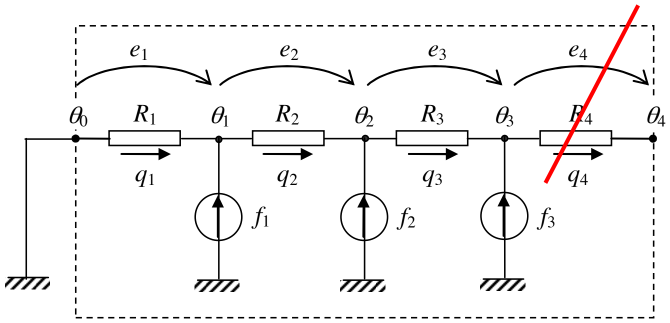
$$\begin{cases} q_1 - q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 - q_4 = -f_3 \end{cases} \qquad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = -\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad \mathbf{f} = -\mathbf{A}^T \cdot \mathbf{q}$$

# Conduction

## Analysis of thermal circuits 1D

- **Dirichlet plus Neumann**





  – circuit equation

$$\mathbf{f} = \mathbf{H} \cdot \boldsymbol{\theta} \qquad \mathbf{G} = \mathbf{I} \qquad \mathbf{H} \equiv \mathbf{A}^{\mathbf{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

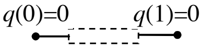
$$\boldsymbol{\theta} = \mathbf{K}^{-1} \cdot \mathbf{f}$$

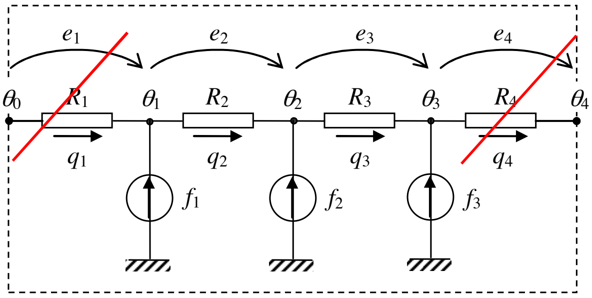
---

# Conduction

## Analysis of thermal circuits 1D

- **Neumann conditions at both boundaries**





  – temperature differences

$$e_1 = 0$$
$$\begin{cases} e_2 = \theta_1 - \theta_2 \\ e_3 = \theta_2 - \theta_3 \end{cases}$$
$$e_4 = 0$$

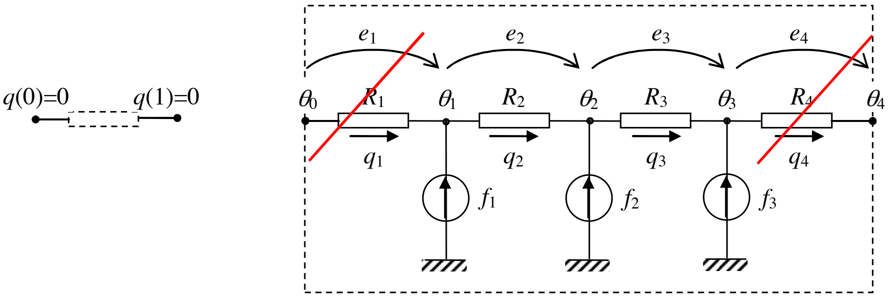
$$\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\mathbf{e} = -\mathbf{A} \cdot \boldsymbol{\theta}$$

# Conduction

## Analysis of thermal circuits 1D

- **Neumann conditions at both boundaries**



  - connectivity matrix

$$\begin{array}{lccccl} & \text{nœud} & \theta_1 & \theta_2 & \theta_3 & \text{arc} \\ & & & & & \cancel{R_1} \\ \mathbf{A} = & & -1 & 1 & 0 & R_2 \\ & & 0 & -1 & 1 & R_3 \\ & & & & & \cancel{R_4} \end{array}$$

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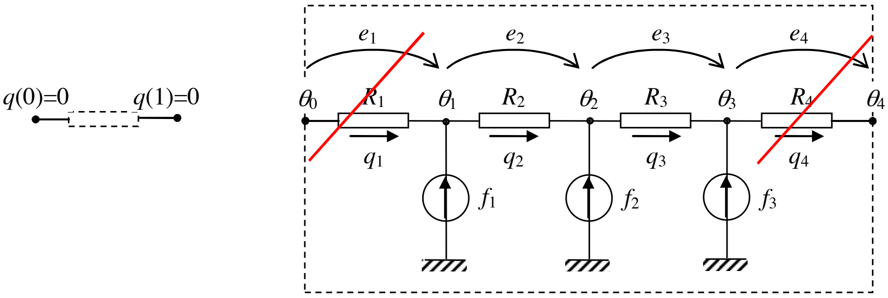
# Conduction

## Analysis of thermal circuits 1D

- **Neumann conditions at both boundaries**



  - constitutive law

$$q_1 = 0$$

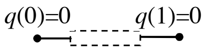
$$\begin{cases} q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \end{cases} \qquad \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} G_2 & 0 \\ 0 & G_3 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

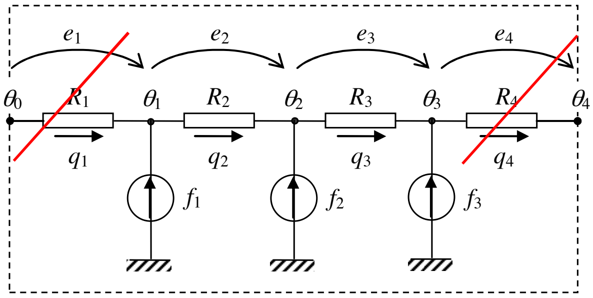
$$q_4 = 0$$

$$G_i = 1 / R_i, i = 1,\ldots,4$$

# Conduction
## Analysis of thermal circuits 1D

- **Neumann conditions at both boundaries**





  - heat balance

$$\begin{cases} -q_2 = -f_1 \\ q_2 - q_3 = -f_2 \\ q_3 = -f_3 \end{cases} \qquad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = -\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} \qquad \mathbf{f} = -\mathbf{A}^T \cdot \mathbf{q}$$

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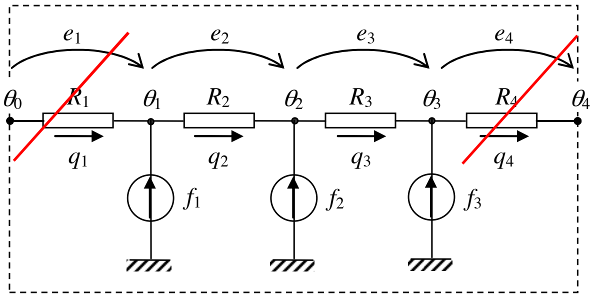
# Conduction
## Analysis of thermal circuits 1D

- **Neumann conditions at both boundaries**



  - Circuit equation

$$\mathbf{f} = \mathbf{B} \cdot \boldsymbol{\theta} \qquad \mathbf{C} = \mathbf{I} \qquad \mathbf{B} \equiv \mathbf{A}^{\mathbf{T}}\mathbf{C}\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

*singulière* ou *non inversable*

To solve: temperature reference is needed

# Conduction

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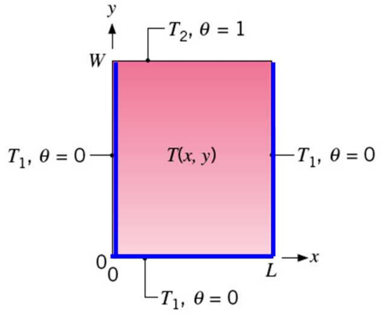
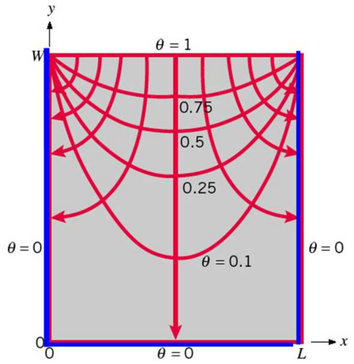
- **2D, steady-state, without internal sources**





$$\rho c \frac{\partial T}{\partial t} = div(\lambda\, \mathbf{grad} T) + p \quad \longrightarrow \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Eq. Laplace

---

# Conduction

## Analysis of thermal circuits 2D

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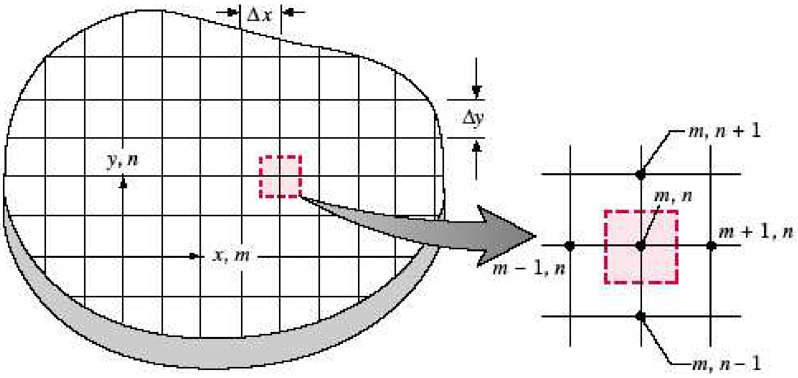
- **Finite differences**
  - Discretization of derivative
- **Finite volume**
  - mean values of conservative variables in a volume, not at the nodes (by integration)
  - non-invasive boundary conditions
  - structured or unstructured meshing
- **Finite element**
  - use a function on a domain
  - unstructured meshing

# Conduction

## Analysis of thermal circuits 2D

- **Finite volume**
  - volume discretization



  - energy balance for every control volum

$$\dot{E}_e - \dot{E}_s + \dot{E}_g = 0$$

---

# Conduction

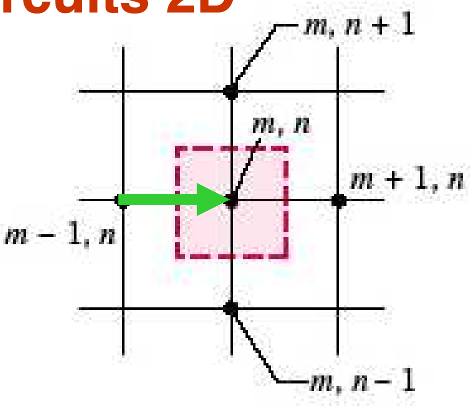
## Analysis of thermal circuits 2D

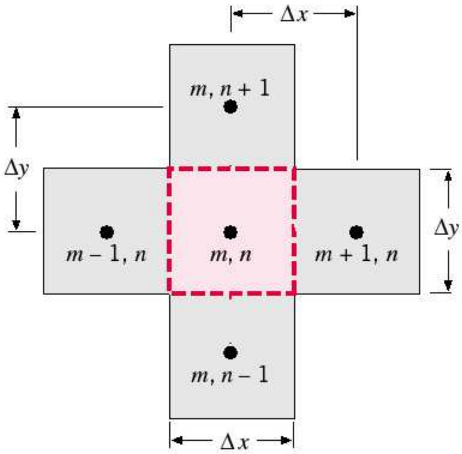
- flux

$$q_x = -\lambda A \frac{dT}{dx}$$

$$q_{(m-1,n)\rightarrow(m,n)} = \lambda\underbrace{(\Delta y \cdot 1)}_{A}\underbrace{\frac{\theta_{m-1,n} - \theta_{m,n}}{\Delta x}}_{-\mathbf{grad}\,\theta}$$

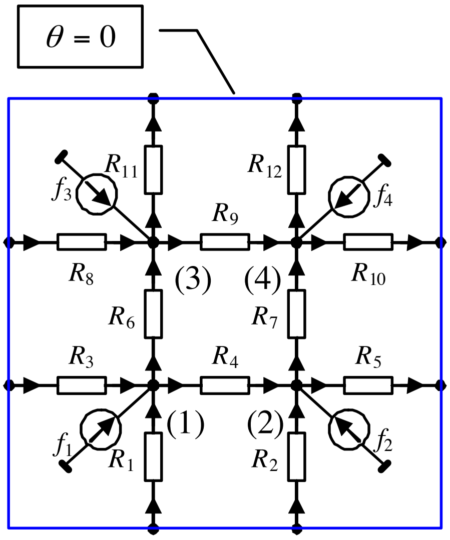
$$R_{i,j} = \frac{1}{\lambda}\frac{L}{A} = \frac{1}{\lambda}\frac{\Delta x}{\Delta y \cdot 1}$$





# Conduction
## Analysis of thermal circuits 2D

- **Dirichlet boundary conditions**



$$\mathbf{A} = \begin{array}{c|cccc|c} \text{nœud} & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \text{arc} \\ \hline & 1 & 0 & 0 & 0 & R_1 \\ & 0 & 1 & 0 & 0 & R_2 \\ & 1 & 0 & 0 & 0 & R_3 \\ & -1 & 1 & 0 & 0 & R_4 \\ & 0 & -1 & 0 & 0 & R_5 \\ & -1 & 0 & 1 & 0 & R_6 \\ & 0 & -1 & 0 & 1 & R_7 \\ & 0 & 0 & 1 & 0 & R_8 \\ & 0 & 0 & -1 & 1 & R_9 \\ & 0 & 0 & 0 & -1 & R_{10} \\ & 0 & 0 & -1 & 0 & R_{11} \\ & 0 & 0 & 0 & -1 & R_{12} \end{array}$$

---

# Conduction
## Analysis of thermal circuits 2D

- **Dirichlet boundary conditions**
  - Circuit equation

$$\mathbf{f} = \mathbf{K} \cdot \boldsymbol{\theta}$$

$$f_i = p(\Delta x \cdot \Delta y \cdot 1) \qquad p\ [\mathrm{W/m^3}]$$

$$\mathbf{K} = \mathbf{A}^T \mathbf{G} \mathbf{A}$$

$$\mathbf{K} = \mathbf{A}^T \mathbf{A} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

# Conduction
## Analysis of thermal circuits 2D

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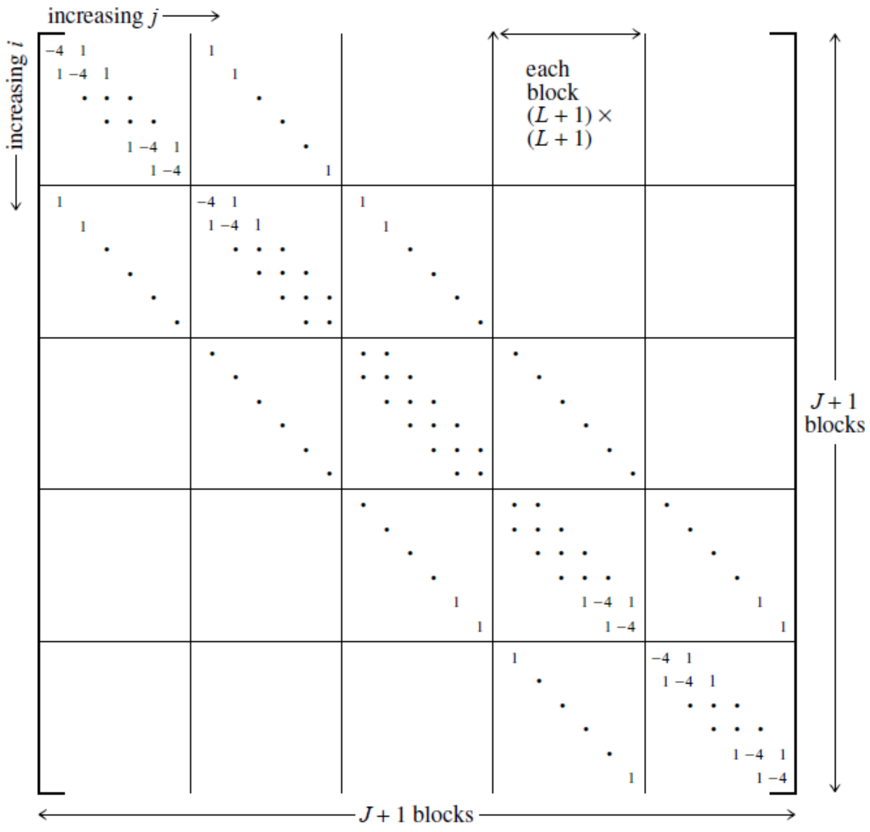

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# Conduction
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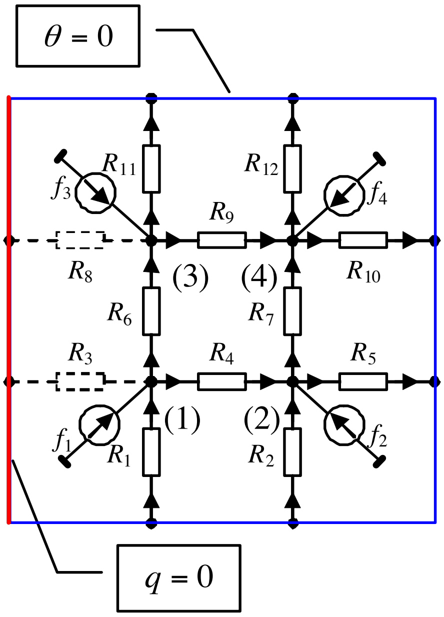
- **Dirichlet and Neumann boundary conditions**



$$
\mathbf{A} = \begin{array}{c|cccc|c}
\text{nœud} & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \text{arc} \\
\hline
 & 1 & 0 & 0 & 0 & R_1 \\
 & 0 & 1 & 0 & 0 & R_2 \\
 & \boxed{0} & 0 & 0 & 0 & R_3 \\
 & -1 & 1 & 0 & 0 & R_4 \\
 & 0 & -1 & 0 & 0 & R_5 \\
 & -1 & 0 & 1 & 0 & R_6 \\
 & 0 & -1 & 0 & 1 & R_7 \\
 & 0 & 0 & \boxed{0} & 0 & R_8 \\
 & 0 & 0 & -1 & 1 & R_9 \\
 & 0 & 0 & 0 & -1 & R_{10} \\
 & 0 & 0 & -1 & 0 & R_{11} \\
 & 0 & 0 & 0 & -1 & R_{12}
\end{array}
$$

# Conduction
## Analysis of thermal circuits 2D

- **Dirichlet and Neumann boundary conditions**
  - circuit equations

$$\mathbf{K} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

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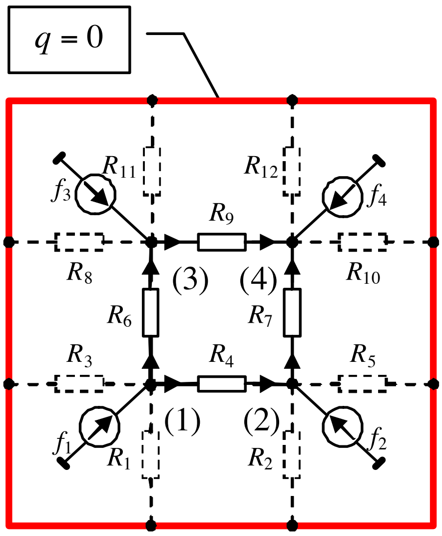
# Conduction
## Analysis of thermal circuits 2D

- **Neumann boundary conditions**



$$\mathbf{A} = \begin{array}{rccccl} \text{nœud} & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \text{arc} \\ & \boxed{0} & 0 & 0 & 0 & R_1 \\ & 0 & \boxed{0} & 0 & 0 & R_2 \\ & \boxed{0} & 0 & 0 & 0 & R_3 \\ & -1 & 1 & 0 & 0 & R_4 \\ & 0 & \boxed{0} & 0 & 0 & R_5 \\ & -1 & 0 & 1 & 0 & R_6 \\ & 0 & -1 & 0 & 1 & R_7 \\ & 0 & 0 & \boxed{0} & 0 & R_8 \\ & 0 & 0 & -1 & 1 & R_9 \\ & 0 & 0 & 0 & \boxed{0} & R_{10} \\ & 0 & 0 & \boxed{0} & 0 & R_{11} \\ & 0 & 0 & 0 & \boxed{0} & R_{12} \end{array}$$

# Conduction
## Analysis of thermal circuits 2D

- **Neumann boundary conditions**

$$\mathbf{K} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

singular matrix

---

# Conduction
## Analysis of thermal circuits 2D

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Dirichlet　　Dirichlet & Neumann　　Neumann

$$\left.\frac{\partial^2\theta}{\partial x^2}\right|_{m,n} = \frac{\left.\dfrac{\partial\theta}{\partial x}\right|_{m+1/2,n} - \left.\dfrac{\partial\theta}{\partial x}\right|_{m-1/2,n}}{\Delta x} = \frac{\theta_{m-1,n} - 2\theta_{m,n} + \theta_{m+1,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2\theta}{\partial y^2}\right|_{m,n} = \frac{\theta_{m,n-1} - 2\theta_{m,n} + \theta_{m,n+1}}{(\Delta y)^2}$$

$$\boxed{-(\Delta x)^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) = 4\theta_{m,n} - \theta_{m,n-1} - \theta_{m,n+1} - \theta_{m-1,n} - \theta_{m+1,n}}$$

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  - Modèle d'état
- Radiation
- Coupled Transfer

- **heat equation**

$$\rho c \frac{\partial T}{\partial t} = -\mathrm{div}(-\lambda\, \mathbf{grad} T) + p$$

- **1D**

$$\rho c \frac{\partial T}{\partial t} = \lambda \frac{d^2 T}{dx^2} + p$$

- **finite differences 1D**

$$\rho \cdot c \dot{\theta}_m = \lambda \frac{1}{\Delta x^2}\left[(\theta_{m-1} - \theta_m) + (\theta_{m+1} - \theta_m)\right] + p$$

$$\rho \cdot A\, \Delta x \cdot c \dot{\theta}_m = \lambda \frac{A}{\Delta x}(\theta_{m-1} - \theta_m) + \lambda \frac{A}{\Delta x}(\theta_{m+1} - \theta_m) + \dot{q} \cdot A\, \Delta x$$

---

# Conduction

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- Coupled Transfer

- **thermal circuit 1D**

$$\rho c \frac{\partial T}{\partial t} = -\mathrm{div}(-\lambda\, \mathbf{grad} T) + p$$

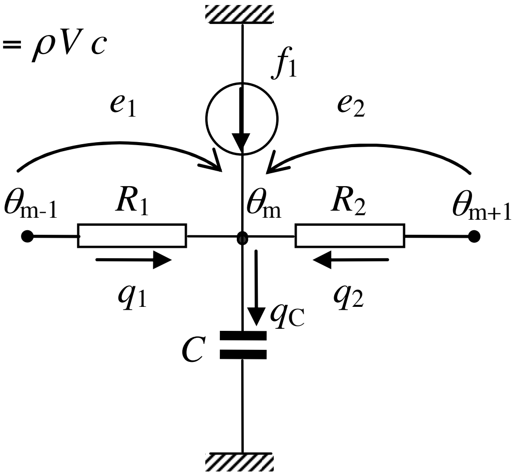
$$\rho \cdot A\, \Delta x \cdot c \dot{\theta}_m = \lambda \frac{A}{\Delta x}(\theta_{m-1} - \theta_m) + \lambda \frac{A}{\Delta x}(\theta_{m+1} - \theta_m) + \dot{q} \cdot A\, \Delta x$$

$$C \equiv \rho\, A\, \Delta x\, c = \rho V c$$

$$R_1 \equiv \lambda \frac{A}{\Delta x}$$

$$R_2 \equiv \lambda \frac{A}{\Delta x}$$

$$f_1 \equiv \dot{q}\, A\, \Delta x$$



# Conduction

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- **het equation**
- **steady-state**

$$0 = -\mathrm{div}(-\lambda\, \mathbf{grad}T) + p$$

$$0 = -\mathbf{A}^T\ \mathbf{G}\ \mathbf{A}\ \boldsymbol{\theta} + \mathbf{f}_T\ ;\quad \mathbf{f}_T \equiv \mathbf{A}^T\ \mathbf{G}\,\mathbf{b} + \mathbf{f}$$

- **dynamic**

$$\rho c \frac{\partial T}{\partial t} = -\mathrm{div}(-\lambda\, \mathbf{grad}T) + p$$

$$\mathbf{C}\dot{\boldsymbol{\theta}} = -\mathbf{A}^T\mathbf{G}\mathbf{A}\cdot\boldsymbol{\theta} + \mathbf{f}_T\ ;\ \mathbf{f}_T \equiv \mathbf{A}^T\mathbf{G}\mathbf{b} + \mathbf{f}$$

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# Conduction

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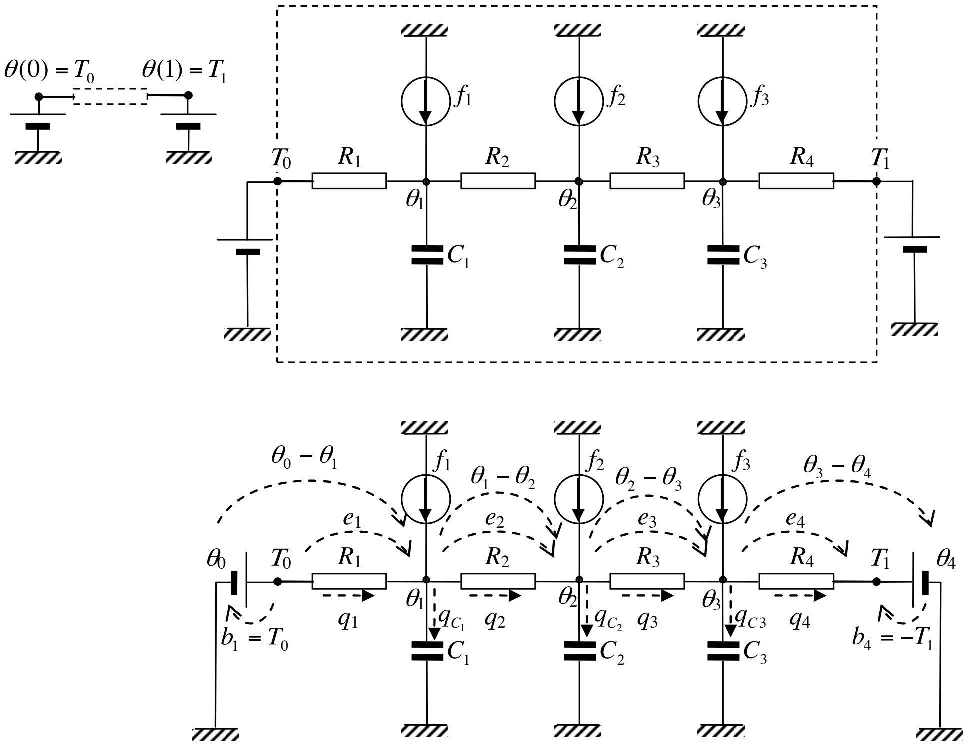

# Conduction
## Analysis of thermal circuits: Dynamic

$$\begin{cases} e_1 - b_1 = \theta_0 - \theta_1 \\ e_2 - b_2 = \theta_1 - \theta_2 \\ e_3 - b_3 = \theta_2 - \theta_3 \\ e_4 - b_4 = \theta_3 - \theta_4 \end{cases} \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} - \begin{bmatrix} b_1 \\ 0 \\ 0 \\ b_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \qquad \mathbf{e} - \mathbf{b} = -\mathbf{A}\boldsymbol{\theta}$$

$$\begin{cases} q_1 = G_1 e_1 \\ q_2 = G_2 e_2 \\ q_3 = G_3 e_3 \\ q_4 = G_4 e_4 \end{cases} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & G_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad \mathbf{q} = \mathbf{G} \cdot \mathbf{e}$$

$$\mathbf{C}\dot{\boldsymbol{\theta}} = \mathbf{A}^T \mathbf{q} + \mathbf{f}$$

$$\begin{cases} C_1\dot{\theta}_1 = q_1 - q_2 + f_1 \\ C_2\dot{\theta}_2 = q_2 - q_3 + f_2 \\ C_3\dot{\theta}_3 = q_3 - q_4 + f_3 \end{cases} \qquad \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\mathbf{C}\,\dot{\boldsymbol{\theta}} = -\mathbf{A}^T\mathbf{G}\,\mathbf{A}\,\boldsymbol{\theta} + \mathbf{A}^T\mathbf{G}\,\mathbf{b} + \mathbf{f} \qquad \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{A} \\ -\mathbf{A}^T & s\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix}$$

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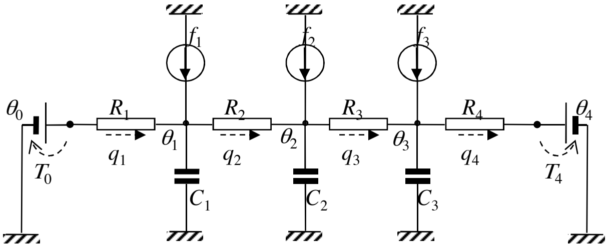
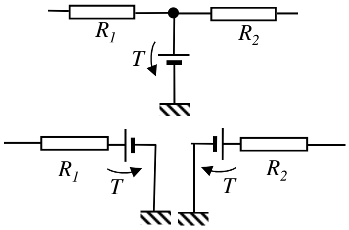
# Conduction
## Analysis of thermal circuits: Dynamic

- **Plug the sources, chose a direction for fluxes**





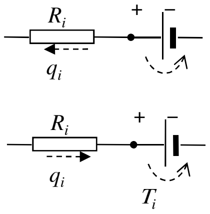
- **Describe the circuit**

$$\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]^T$$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} G_1 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & G_4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} T_0 \\ 0 \\ 0 \\ -T_4 \end{bmatrix}$$

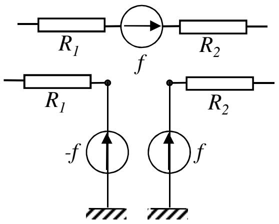
$$\mathbf{C} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}$$

$$\mathbf{f} = [f_1 \quad f_2 \quad f_3]^T$$



$$b_i = T_i$$

$$b_i = -T_i$$



- **Give the solution**

$$\mathbf{C}\dot{\boldsymbol{\theta}} = -\mathbf{A}^T\mathbf{G}\mathbf{A} \cdot \boldsymbol{\theta} + \mathbf{f}_T \; ; \quad where \ \mathbf{f}_T \equiv \mathbf{A}^T\mathbf{G} \cdot \mathbf{b} + \mathbf{f}$$

# Conduction

## Analysis of thermal circuits: Dynamic

- **state-space model, if** $\exists\mathrm{C}^{-1}$

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_s\,\boldsymbol{\theta} + \mathbf{B}_s\,\mathbf{f}_T$$
$$\mathbf{y} = \mathbf{C}_s\,\boldsymbol{\theta} + \mathbf{D}_s\,\mathbf{f}_T$$

where $\boldsymbol{\theta}$ state vector; contains the temperatures

$\mathbf{f}_T = [\mathbf{b} \quad \mathbf{f}]^T$ - input vector; contains sources

$\mathbf{y}$ - output vector; varaibles of interest,

$\mathbf{A}_s = -\mathbf{C}^{-1}\mathbf{A}^T\mathbf{G}\mathbf{A}$ dynamc matrix, if $\exists\mathbf{C}^{-1}$

$\mathbf{B}_s = \mathbf{C}^{-1}[\mathbf{A}^T\mathbf{G} \quad \mathbf{I}]$ - command matrix, if $\exists\mathbf{C}^{-1}$

$\mathbf{C}_s$ - observation matrix

$\mathbf{D}_s$ - feed-through matrix

---

# Conduction

## Analysis of thermal circuits: Dynamic

- **Euler approximation**

$$\left.\frac{\partial\theta}{\partial t}\right|_{m,n} \cong \frac{\theta_{m,n}^{p+1} - \theta_{m,n}^{p}}{\Delta t}$$

- **evalation at he previous time-step**

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_s\,\boldsymbol{\theta} + \mathbf{B}_s\,\mathbf{f}_T$$

$$\frac{\boldsymbol{\theta}_{p+1} - \boldsymbol{\theta}_p}{\Delta t} = \mathbf{A}_s\,\boldsymbol{\theta}_p + \mathbf{B}_s\,\mathbf{f}_{T,p}$$

$$\boldsymbol{\theta}_{p+1} = (\mathbf{I} + \Delta t\,\mathbf{A})\,\boldsymbol{\theta}_p + \Delta t\,\mathbf{B}_s\,\mathbf{f}_{T,p}$$

# Conduction

## Analysis of thermal circuits: Dynamic

- **Stability**
  - choice of the spatial discretization
  - imposes the time-step

- *1D :*

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

*2D :*

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{4}$$

$$\alpha = \lambda / (\rho c)$$ thermal diffusivity

---

# Conduction

## Analysis of thermal circuits: Dynamic

- **Euler approximation**

$$\left.\frac{\partial \theta}{\partial t}\right|_{m,n} \cong \frac{\theta_{m,n}^{p+1} - \theta_{m,n}^{p}}{\Delta t}$$

- **evaluation at the same time-step**

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_s \boldsymbol{\theta} + \mathbf{B}_s \mathbf{f}_T$$

$$\frac{\boldsymbol{\theta}_{p+1} - \boldsymbol{\theta}_p}{\Delta t} = \mathbf{A}_s \boldsymbol{\theta}_{p+1} + \mathbf{B}_s \mathbf{f}_{T,p}$$

$$\boldsymbol{\theta}_{p+1} = (\mathbf{I} - \Delta t \mathbf{A})^{-1} (\boldsymbol{\theta}_p + \Delta t \, \mathbf{B}_s \mathbf{f}_{T,p})$$

# Convection

## Constitutive law: Newton

- **Heat transfer in fluids**
  - *advection*
  - *diffusion*

- **Steady-state**
- **Fluid – solid wall**
  - heat flux at the wall

$$\varphi_p = -\lambda_{ps} \left.\frac{\partial \theta}{\partial n}\right|_{ps} = -\lambda_{pf} \left.\frac{\partial \theta}{\partial n}\right|_{pf}$$

  - temperature continuity at the wall

$$\theta_s\big|_M = \theta_f\big|_M$$

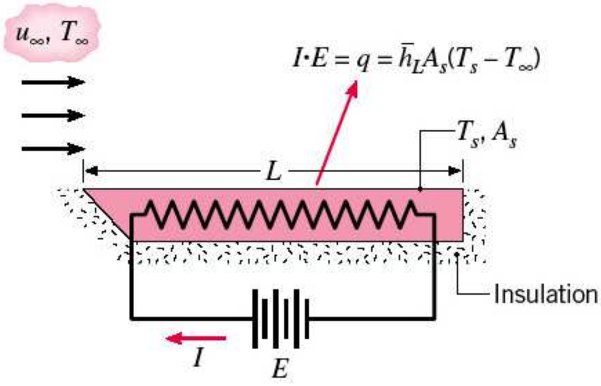
---

# Convection

## Constitutive law: Newton

- **Newton law**

$$\varphi_p = h_c(\theta_p - \theta_M)$$



- **Convection problem**
  - equivalent temperature
  - convective heat coefficient

# Convection
## Constitutive law: Newton

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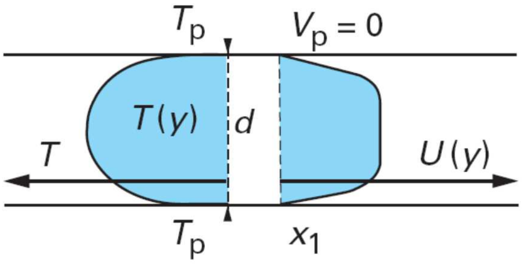
- **Internal flow**



$$\dot{m}_x = \int_0^d \rho\, v\, dy = d\ \rho\, v_M \qquad v_M \equiv \frac{1}{d}\int_0^d v\, dy$$

$$\dot{H}_x = \int_0^d \rho\, c\, v\, \theta\, dy \qquad \theta_M = \frac{\dot{H}_x}{c\, \dot{m}_x} = \frac{\int_0^d \rho\, v\, \theta\, dy}{\int_0^d \rho\, v\, dy}$$

---

# Convection
## Characteristic temperature

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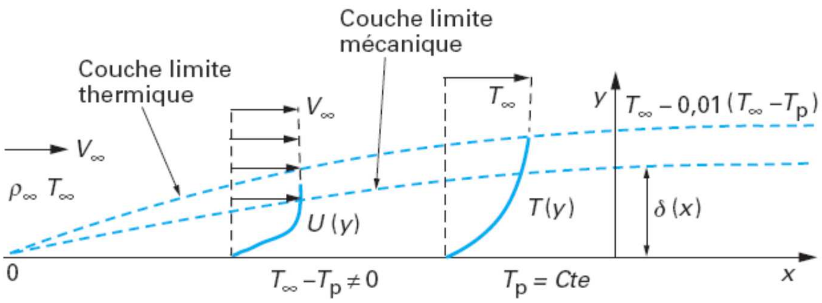
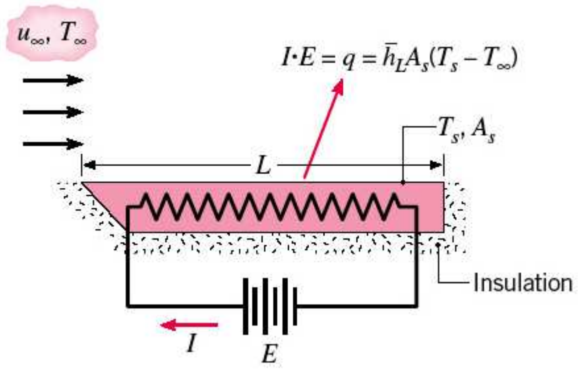
- **External flow**

$$\theta_M = \theta_\infty$$





# Convection
## Heat rate coefficient

- **Approximative values**

| | Convection | |
|---|---|---|
| | natural (W/m$^2$ K) | forced (W/m$^2$ K) |
| gaz | 3 …30 | 12 … 200 |
| liquid | 30 …300 | 200…7500 |

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# Convection
## Heat rate coefficient

- **Mathematical expression of a physical law**

$$\begin{tikzcd} f(E_1,\dots,E_p)=0 \arrow[d, "\text{Theorem of Vachy – Buckingham}"] \\ F(\pi_1,\dots,\pi_{p-q})=0 \end{tikzcd}$$

$q$ number of fundamental units

$(L, T, M, I, \dots)$

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## Heat rate coefficient

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- **Example : heat exchange in an infinite pipe**

Values which influence the thermal flux in the pipe

| | Value | Symbol | SI unit | Dimension |
|---|---|---|---|---|
| 1 | Pipe diameter | (*a*) $D$ | m | $[L]$ |
| 2 | Fluid velocity | (*b*) $U_\infty$ | m/s | $[LT^{-1}]$ |
| 3 | Fluid density | (*c*) $\rho$ | $\mathrm{kg/m^3}$ | $[ML^{-3}]$ |
| 4 | Fluid viscosity | (*d*) $\mu$ | $kg\ /(m \cdot s)$ | $[ML^{-1}T^{-1}]$ |
| 5 | Thermal conductivity of the fluid | (*e*) $\lambda$ | $\mathrm{W/(m \cdot K)}$ | $[MLT^{-3}\theta^{-1}]$ |
| 6 | Specific heat at constant pressure | (*f*) $c_p$ | $\mathrm{J/(kg \cdot K)}$ | $[L^2T^{-2}\theta^{-1}]$ |
| 7 | Heat convection coefficient | (*g*) $h_c$ | $\mathrm{W/(m^2K)}$ | $[MT^{-3}\theta^{-1}]$ |

7 [physical variables] - 4 [dimensions ($L$, $M$, $T$, $\theta$)] = 3 [products]

$$\pi = D^a \cdot v^b \cdot \rho^c \cdot \mu^d \cdot \lambda^e \cdot c_p^f \cdot h_c^g$$

dimensions :

$$[\pi] = [L]^{a+b-3c-d+e+2f}[M]^{c+d+e+g}[T]^{-b-d-3e-2f-3g}[\theta]^{-e-f-g}$$

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– adimensionel products (4 eqs. 7 unknowns)

$$\begin{cases} a + b - 3c - d + e + 2f = 0 \\ c + d + e + g = 0 \\ -b - d - 3e - 2f - 3g = 0 \\ -e - f - g = 0 \end{cases}$$

– a possible solution

$$a = 1; e = -1; g = 1; b = c = d = f = 0; \Rightarrow \pi_1 = \frac{h \cdot D}{\lambda} \equiv Nu$$

$$a = 1; b = 1; c = 1; d = -1; e = f = g = 0; \Rightarrow \pi_2 = \frac{\rho \cdot v \cdot D}{\mu} \equiv Re$$

$$d = 1; e = -1; f = 1; a = b = c = g = 0; \Rightarrow \pi_3 = \frac{\mu \cdot c_p}{\lambda} \equiv Pr$$

$$F(Nu, Re, Pr) = 0$$

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- **Reynolds** : ratio between inertial and viscous forces

$$Re = \frac{\rho v D}{\mu} = \frac{\rho v^2 D^2}{\mu v D}$$

- *inertia*

$$\rho v^2 D^2 = \rho v D^2 \cdot v = \rho l D^2 \cdot \frac{v}{t} \propto \rho V \cdot \frac{v}{t} = m \cdot a$$

- *forces viscous*

$$\mu v D \propto \mu S \frac{dv}{dz}$$

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- **Prandtl** : viscous diffusion rate / thermal difussion rate

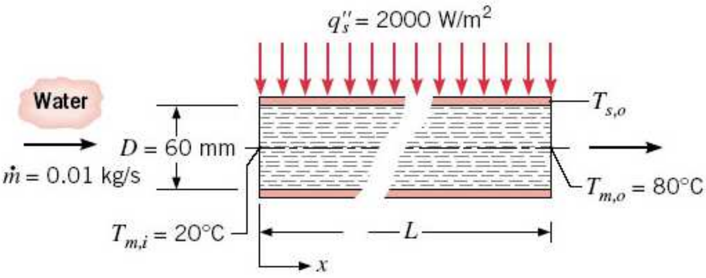
$$Pr = \frac{\mu \cdot c_p}{\lambda} = \frac{\mu / \rho}{\lambda / (\rho \cdot c_p)} = \frac{\nu}{\alpha}$$

- **Nusselt** : convective het transfer / conductive het transfer

$$Nu = \frac{hD}{\lambda} = \frac{h \Delta T}{\lambda \frac{\Delta T}{D}} = \frac{D / \lambda}{1 / h}$$

# Convection
## Heat rate coefficient

- **inside flow**



$h_c$ depends on:

- $v_M$ , mean velocity of the fluid, m/s
- $\rho$ , density of fluid, kg/m$^3$
- $c$ , specific heat of fluid, J/kg °C
- $\mu$ , dynamic viscosity of fluid, Pa s
- $\lambda$ , thermal conductifvity of fluid, W/m °C
- $D$ , inner diameter of pipe, m
- $x$ , abscissa, m.

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# Convection
## Heat rate coefficient

- **internal flow**

Four fundamental units (four adimensinal numbers):

$Nu = \frac{h\,D}{\lambda}$ Nusselt, het exchange between fluid and wall;

$\mathrm{Re} = \frac{\rho\, v_m\, D}{\mu}$ Reynolds, flow type:

$Re < 2000$ laminair flow
$Re > 3000$ turbulent flow

$Pr = \frac{\mu\, c}{\lambda}$ Prandtl, thermal properties of the fluid

$\frac{x}{D}$ adimensional abscissa

$$Nu = 0.023\,\mathrm{Re}^{0.8}\ \mathrm{Pr}^{0.33}$$

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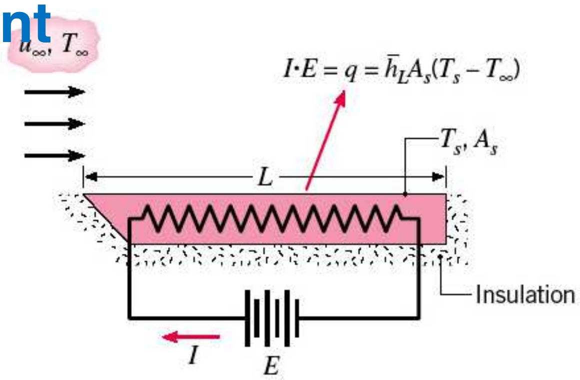
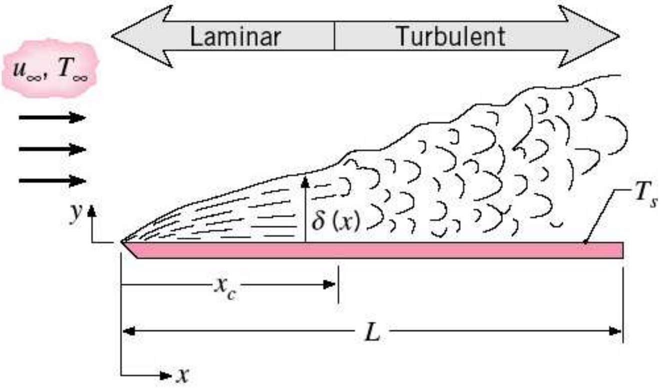
- **external flow**



$$\overline{Nu} = 0.035 \, \mathrm{Re}^{0.80} \ \mathrm{Pr}^{0.33}$$



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- **Gravity and buoyancy**
- **Grashof number: equivalent to Raynolds number for natural convection**

$$Gr \equiv \frac{g \rho \beta (\theta_p - \theta_\infty) l^3}{\mu^2}$$

  - buoyancy (Archimedes)
  - viscous forces

- **Correlation for natural convection**

$$Nu = C(Gr \ Pr)^n \qquad Ra \equiv Gr \ Pr \, (\text{Rayleigh})$$