Building design: multidisciplinary Approach

Structural part

Optional Lecture
GCU - S8 – M8
Civil Engineering and Urban Planning Department

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Outline

Outline

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Introduction

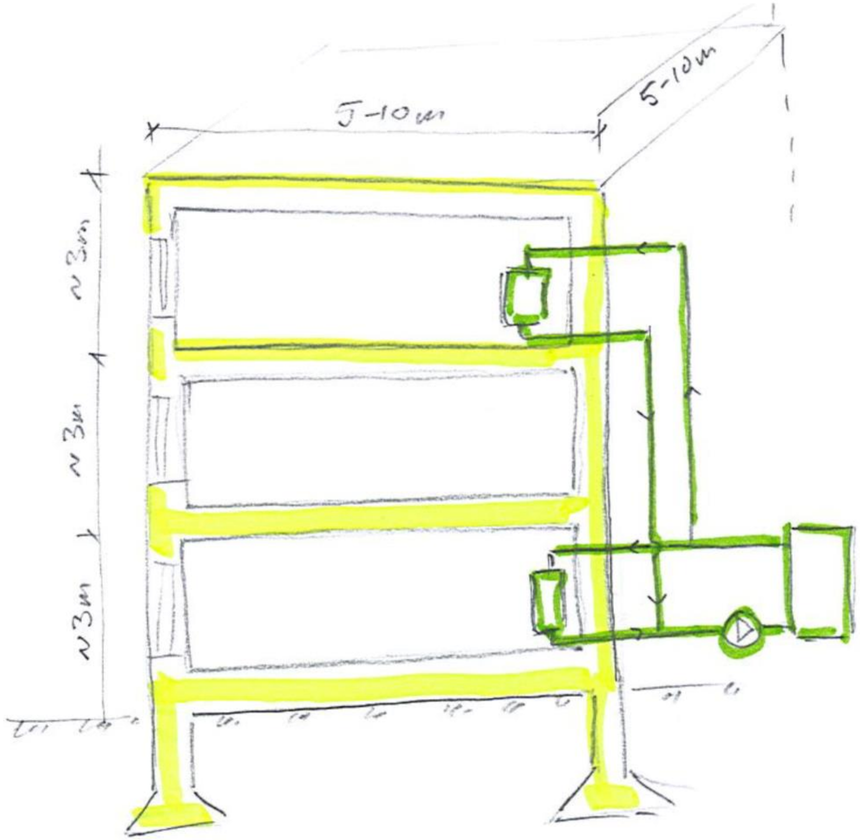
Design a simple building accounting for **several domains**

⇒ Hydraulic

⇒ Energetic

⇒ Geotechnic

⇒ Structure



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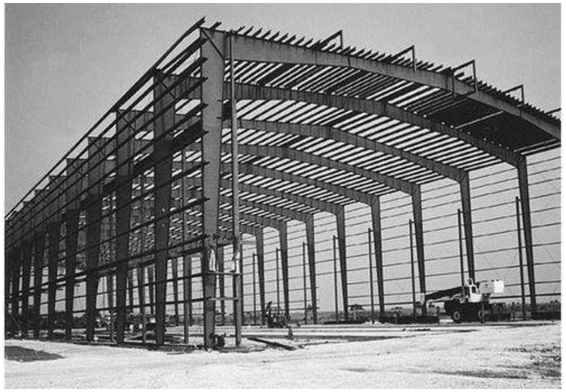
Introduction



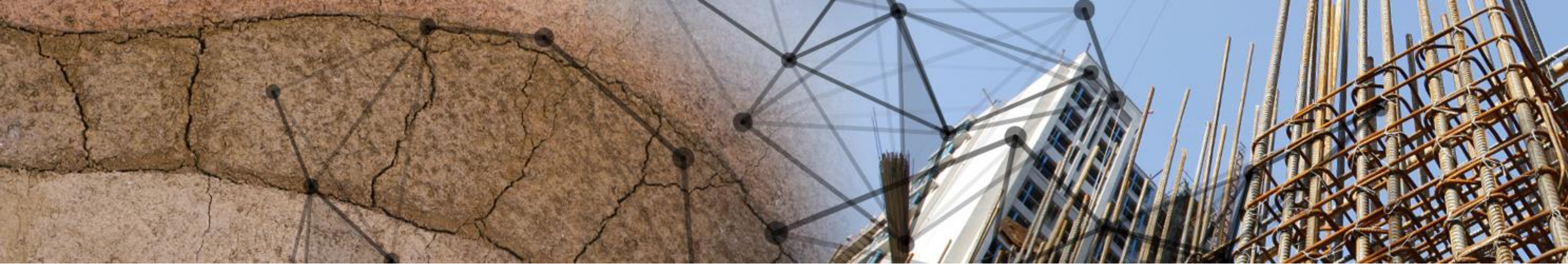
Focus on frame structures

Objectives

⇒ Design based on **EuroCodes** criteria (see first semester)
⇒ **Material** up to the students (steel, RC, wood…)
⇒ **Own computational code development** (MATLAB)

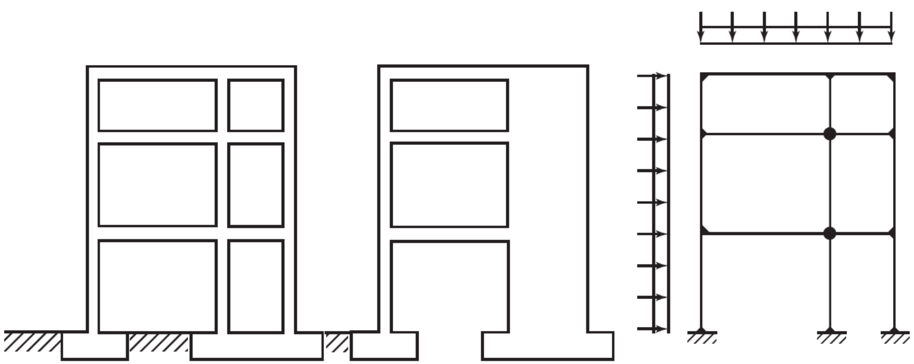
Results

⇒ Propose **an integrated design** of a simple building

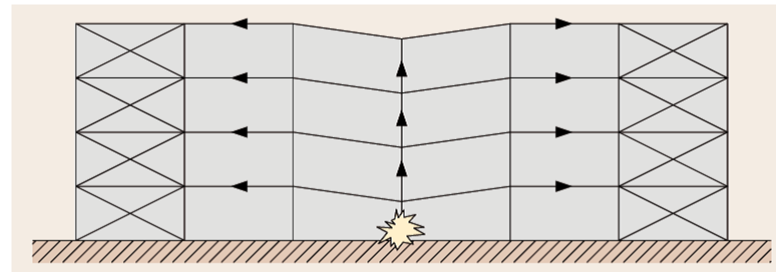


Introduction

Focus on 2D frames : examples



(a) Sway frame (b) Non-sway frame (c) Idealized frame



Progressive collapse

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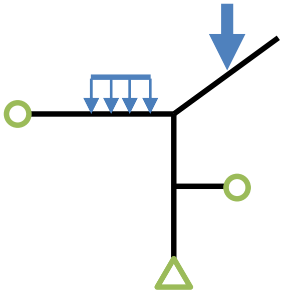
Introduction : *Limits states*

- Ultimate (strength)
- Service (deformability)

EC criteria depends on:

- Geometry
- Material

INPUTS
- Geometry
- Material
- Loading
- Boundary conditions,etc.



FE model (Type of analysis)
- Linear elastic
- QS / Dynamic
- Small/large defo.
- *etc.*

SLS analysis
- Displacment field (local/global)
- Modal analysis
- *etc.*

ULS analysis
- MNT maxima
- Cross section dimensions
- *etc.*

Introduction : *Scope of the project*

Main hypothesis

- **2D** frames (post-beam structures)
- **Linear elastic** analysis (applied for steel, RC, wood, *etc.*)
- **Quasi-static** and **dynamic** loadings (Earthquakes, impacts)

Design tool

- **Finite Element** Analysis (home made MATLAB code)

Results

- **Displacement** field
- Internal **efforts**
- **Cross-section** definition

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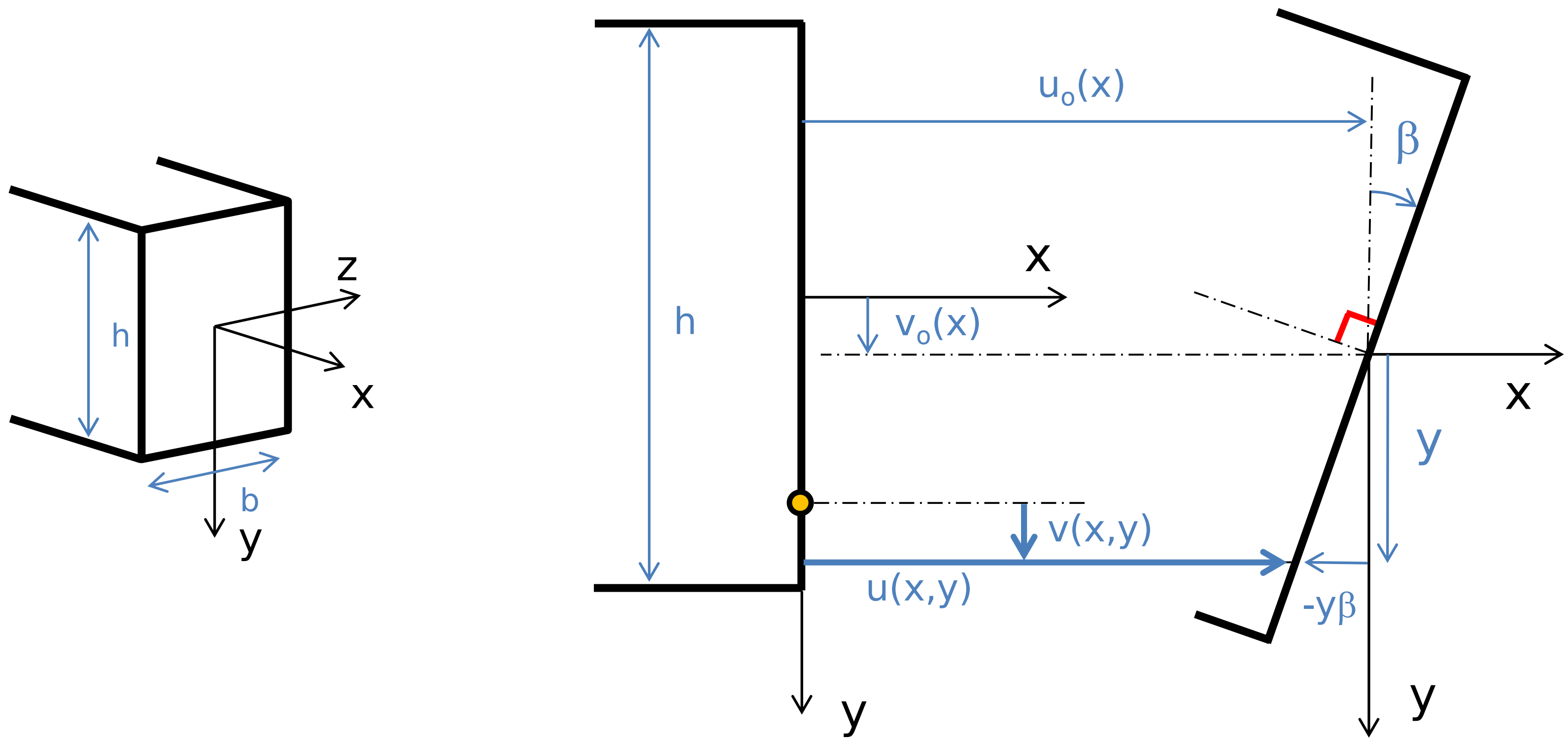
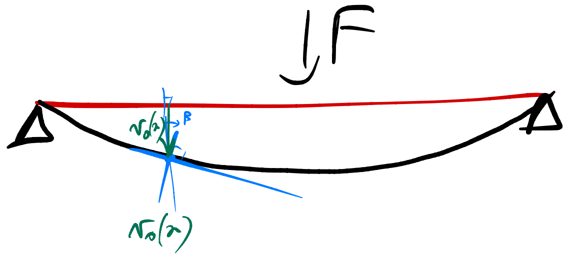
FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D



with h >> $u_o(x)$ and $v_o(x)$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FEA in *quasi*-static conditions

$\Rightarrow$ Formulation of Euler-Bernoulli beam FE in 3D

3D

$$\begin{cases} u(x,y,z) = u_o(x) - y\frac{dv_o}{dx} - z\frac{dw_o}{dx} \\ v(x,y,z) = v_o(x) - z\theta_x(x) \\ w(x,y,z) = w_o(x) + y\theta_x(x) \end{cases}$$

Displacement field

EB kinematic assumptions

$$\begin{cases} \epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_o}{dx} - y\frac{d^2 v_o}{dx^2} - z\frac{d^2 w_o}{dx^2} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\frac{d\theta_x}{dx} \\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = y\frac{d\theta_x}{dx} \end{cases}$$

Strain tensor

$$\beta = \frac{\partial v_0}{\partial x}$$

$$\sigma_{yy} = \sigma_{zz} = \tau_{yz} = 0 \text{ are supposed null}$$

In the framework of EB beam theory

$$\begin{cases} \sigma_{xx} = E\epsilon_{xx} \\ \tau_{xy} = G\gamma_{xy} \\ \tau_{xz} = G\gamma_{xz} \end{cases}$$

Stress tensor

Distorsions due to torsion

$\rightarrow$ We stay in 2D

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FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

2D

$$\begin{cases} u(x,y\quad) = u_o(x) - y\frac{dv_o}{dx} \\ v(x,y\quad) = v_o(x) \end{cases}$$

Displacement field
EB kinematic assumptions

$$\begin{cases} \epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_o}{dx} - y\frac{d^2 v_o}{dx^2} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \end{cases}$$

Strain tensor

$$\beta = \frac{\partial v_0}{\partial x}$$

$\sigma_{yy} =$ $\tau_{yz} = 0$ are supposed null

In the framework of EB beam theory

$$\begin{cases} \sigma_{xx} = E\epsilon_{xx} \\ \tau_{xy} = \text{Calculated from cross section balance} \end{cases}$$

Stress tensor

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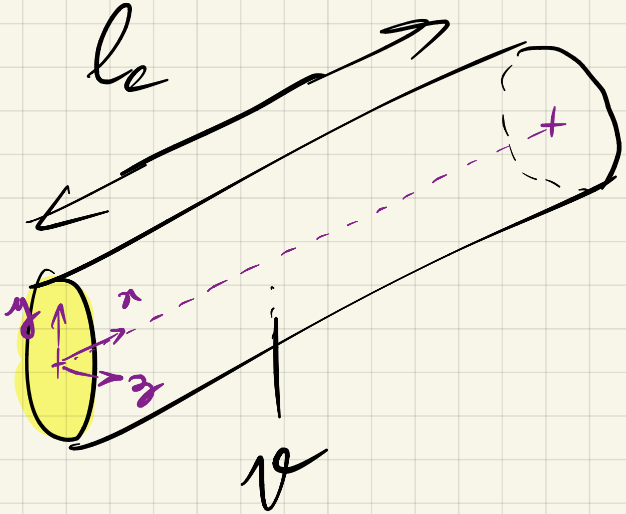
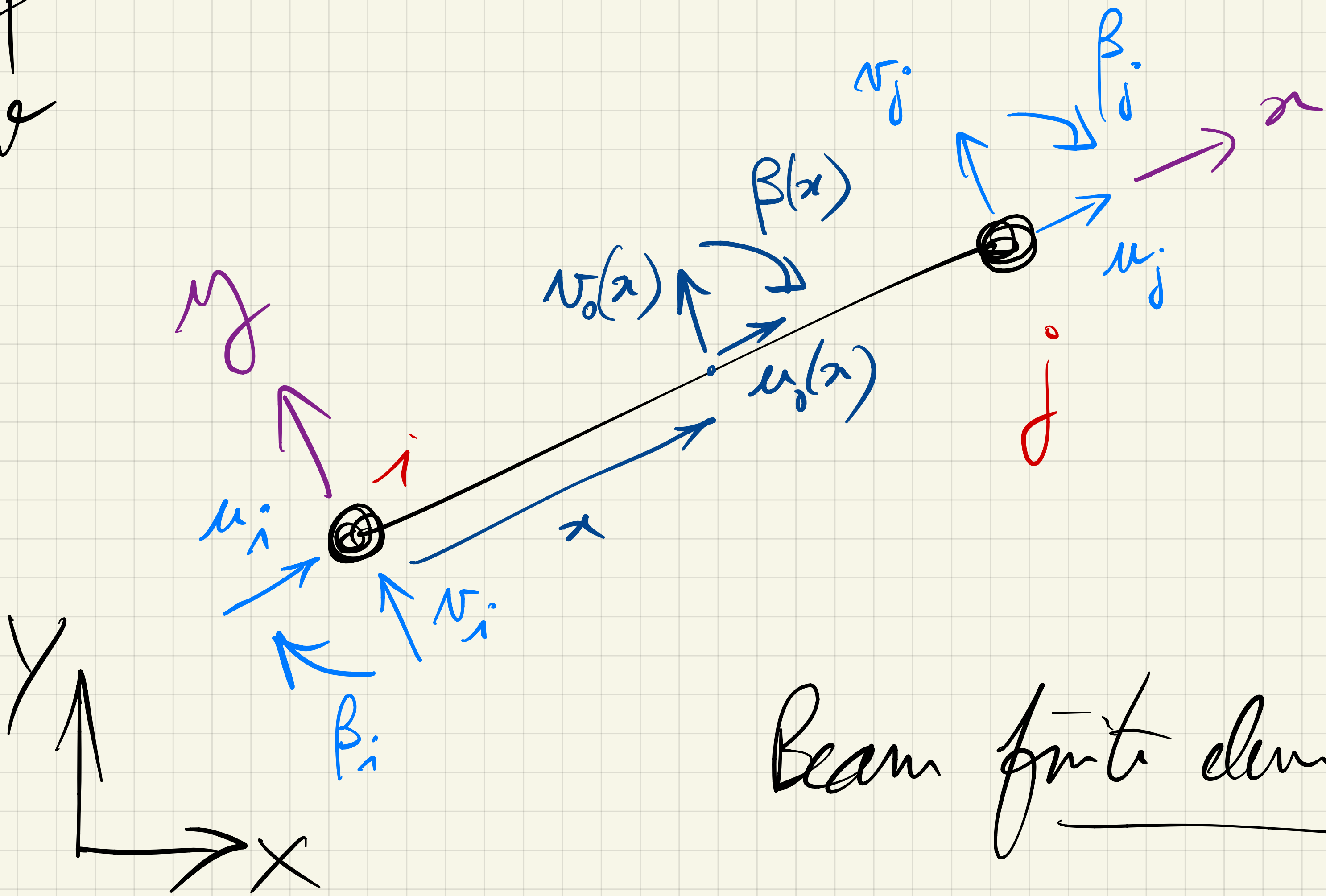
Beam theory under the kinematic assumption



$$\begin{bmatrix} u_0(x) \\ v_0(x) \\ \beta(x) \end{bmatrix} = N\,[q_e]$$



Beam finite element

$$u_0(x) = N_1(x)\,u_1 + N_2(x)\,u_2$$

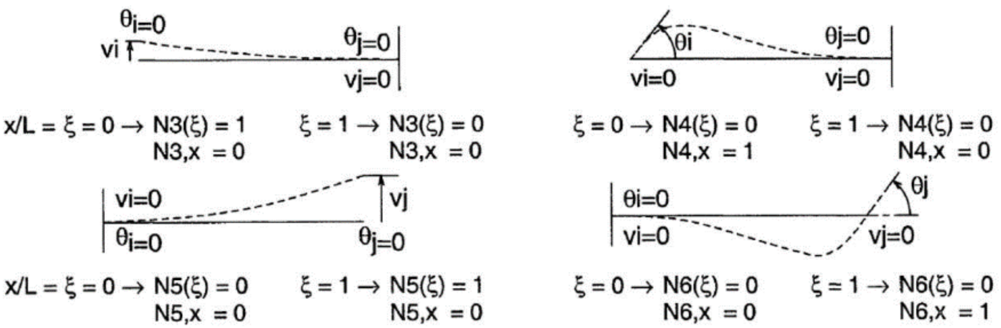
$$v_0(x) = H_1(x)\,v_1 + H_2(x)\,\beta_1 + H_3(x)\,v_2 + H_4(x)\,\beta_2$$

$$\beta(x) = \frac{dv_0}{dx} = \frac{\partial H_1}{\partial x} v_1 + \frac{\partial H_2}{\partial x} \beta_1 + \frac{\partial H_3}{\partial x} v_2 + \frac{\partial H_4}{\partial x} \beta_2$$

→ because $\beta = \dfrac{\partial v_0}{\partial x}$ Euler-Bernoulli kinematic assumption.

FEA in *quasi*-static conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D



$$
\begin{cases}
N_1(x) = 1 - \frac{x}{L_e} \\
H_1(x) = 1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3} \\
H_2(x) = x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2} \\
N_2(x) = \frac{x}{L_e} \\
H_3(x) = \frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3} \\
H_4(x) = -\frac{x^2}{L_e} - \frac{x^3}{L_e^2}
\end{cases}
$$

with the associated derivatives (first and second)

Shape functions

$$
\begin{cases}
N_1'(x) = -\frac{1}{L_e} \\
H_1'(x) = -\frac{6x}{L_e^2} + \frac{6x^2}{L_e^3} \\
H_2'(x) = 1 - \frac{4x}{L_e} + \frac{3x^2}{L_e^2} \\
N_2'(x) = \frac{1}{L_e} \\
H_3'(x) = \frac{6x}{L_e^2} - \frac{6x^2}{L_e^3} \\
H_4'(x) = -\frac{2x}{L_e} + \frac{3x^2}{L_e^2}
\end{cases}
\qquad
\begin{cases}
N_1''(x) = 0 \\
H_1''(x) = -\frac{6}{L_e^2} + \frac{12x}{L_e^3} \\
H_2''(x) = -\frac{4}{L_e} + \frac{6x}{L_e^2} \\
N_2''(x) = 0 \\
H_3''(x) = \frac{6}{L_e^2} - \frac{12x}{L_e^3} \\
H_4''(x) = -\frac{2}{L_e} + \frac{6x}{L_e^2}
\end{cases}
$$

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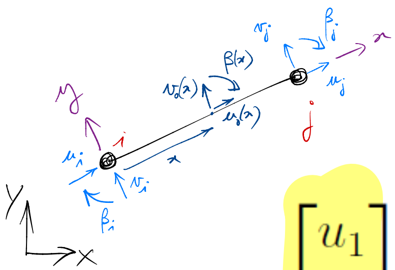
FEA in *quasi*-static conditions

$\Rightarrow$ **Formulation of Euler-Bernoulli beam FE in 2D**



Displacement field interpolation

$$\begin{bmatrix} u_0(x) \\ v_0(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} N_1(x) & 0 & 0 & N_2(x) & 0 & 0 \\ 0 & H_1(x) & H_2(x) & 0 & H_3(x) & H_4(x) \\ 0 & \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial x} & 0 & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \beta_1 \\ u_2 \\ v_2 \\ \beta_2 \end{bmatrix}$$

where

$$\begin{cases} q_e^T = [u_1\ v_1\ \beta_1\ u_2\ v_2\ \beta_2] \\ \mathbb{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & H_1 & H_2 & 0 & H_3 & H_4 \\ 0 & H_1' & H_2' & 0 & H_3' & H_4' \end{bmatrix} \end{cases}$$

$$\beta = \frac{\partial v_0}{\partial x}$$

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FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

Principle of Virtual Work

General from (Volumic integration)

$$\int_V (\epsilon^\star)^T \sigma dV = \int_V (u^\star)^T f_b \, dV + \int_{\partial V} (u^\star)^T F_s \, ds$$

For beams (Lineic integration)

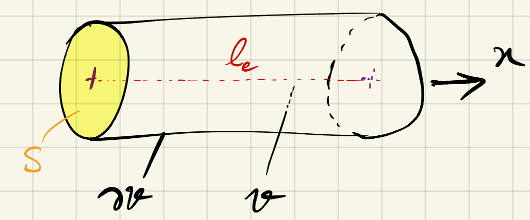
$$\int_\Gamma (\epsilon_b^\star)^T \sigma_b d\Gamma = \int_\Gamma (u_b^\star)^T f_v \, d\Gamma + \sum_{\partial\Gamma} (u_b^\star)^T F_{bs}$$

where Γ is the longitudinal abscissa of the structure

f_v the body forces expressed in N/ml

F_{bs} are the punctual forces

$$\int_V (\varepsilon^*)^T \sigma \, dV = \int_V (u^*)^T f_b \, dV + \int_{\partial V} (u^*)^T F_S \, ds$$



first term : $\displaystyle \int_0^{l_e} \left(\int_S \varepsilon_{xx}^* \cdot \sigma_{xx} \, dS \right) dx$ where $\begin{cases} \varepsilon_{xx} = \dfrac{\partial u_0}{\partial x} - y \dfrac{\partial^2 v_0}{\partial x^2} \\ \sigma_{xx} = E \, \varepsilon_{xx} \end{cases}$

$$\int_S \left(\frac{\partial u_0^*}{\partial x} - y \frac{\partial^2 v_0^*}{\partial x^2} \right) E \left(\frac{\partial u_0}{\partial x} - y \frac{\partial^2 v_0}{\partial x^2} \right) dS$$

$$= E \left[\int_S \frac{\partial u_0^*}{\partial x} \frac{\partial u_0}{\partial x} \, ds - \int_S \frac{\partial u_0^*}{\partial x} \frac{\partial^2 v_0}{\partial x^2} y \, dS - \int_S \frac{\partial^2 v_0^*}{\partial x^2} \frac{\partial u_0}{\partial x} y \, dS + \int_S \frac{\partial^2 v_0}{\partial x^2} \frac{\partial^2 v_0^*}{\partial x^2} y^2 \, dS \right]$$

$$= \frac{\partial u_0^*}{\partial x} \frac{\partial u_0}{\partial x} E \underbrace{\int_S ds}_{=S} - \frac{\partial u_0^*}{\partial x} \frac{\partial^2 v_0}{\partial x^2} E \underbrace{\int_S y \, dS}_{=0} - \frac{\partial^2 v_0^*}{\partial x^2} \frac{\partial u_0}{\partial x} E \underbrace{\int_S y \, dS}_{=0} + \frac{\partial^2 v_0}{\partial x^2} \frac{\partial^2 v_0^*}{\partial x^2} E \underbrace{\int_S y^2 \, dS}_{I_{Gz}}$$

$$= ES \frac{\partial u_0^*}{\partial x} \frac{\partial u_0}{\partial x} + E I_{Gz} \frac{\partial^2 v_0^*}{\partial x^2} \frac{\partial^2 v_0}{\partial x^2}$$

$$\int_0^{l_e}\left(\int_S \varepsilon^{*}_{xx}\cdot\sigma_{xx}\, dS\right)dx = \int_0^{l_e} ES\,\frac{\partial u_0^{*}}{\partial x}\frac{\partial u_0}{\partial x} + EI_{Gz}\frac{\partial^2 v_0^{*}}{\partial x^2}\frac{\partial^2 v_0}{\partial x^2}\; dx$$

Let's define $\sigma_b = \begin{bmatrix} N \\ M \end{bmatrix}$ $\varepsilon_b = \begin{bmatrix} \varepsilon_0 \\ \chi \end{bmatrix}$ where $\begin{cases} \varepsilon_0 = \partial u_0/\partial x \\ \chi = \partial^2 v_0/\partial x^2 \end{cases}$

$$\Rightarrow \int_0^{l_e} \underbrace{\left[\frac{\partial u_0^{*}}{\partial x} \quad \frac{\partial^2 v_0^{*}}{\partial x^2}\right]}_{(\varepsilon_b^{*})^T} \underbrace{\underbrace{\begin{bmatrix} ES & 0 \\ 0 & EI \end{bmatrix}}_{= \mathbb{C}} \underbrace{\begin{bmatrix} \partial u_0/\partial x \\ \partial^2 v_0/\partial x^2 \end{bmatrix}}_{\varepsilon_b}}_{= \sigma_b = \mathbb{C}\,\varepsilon_b} dx$$

$\mathbb{C}$ is the behaviour matrix describing the relation between Static and Kinematic fields

Thus $\displaystyle\int_{\mathcal{V}} (\varepsilon^{*})^T \sigma \, d\mathcal{V} = \int_{\Gamma} (\varepsilon_b^{*})^T \sigma_b \, dx$

Same demo for the second member of the equation with $u_b = \begin{bmatrix} u_0 \\ v_0 \\ \beta \end{bmatrix}$

FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

Behavior law (elastic linear material)

stress and strain tensors (generalized version[1])

$$\sigma_b = \mathbb{C}\epsilon_b \Leftrightarrow \begin{bmatrix} N \\ M(x) \end{bmatrix} = \mathbb{C} \begin{bmatrix} \epsilon_0 \\ \chi(x) \end{bmatrix}$$

where ϵ_0 is the axial strain
χ is the beam curvature

$$\mathbb{C} = \begin{bmatrix} ES & 0 \\ 0 & EI \end{bmatrix}$$ **Elastic mat.**

E is the *Young* modulus
S the cross section area
I its inertia

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FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

Remainder

$$q_e = \begin{bmatrix} u_1 \\ v_1 \\ \beta_1 \\ u_2 \\ v_2 \\ \beta_2 \end{bmatrix}$$

$$\epsilon_b = \begin{bmatrix} \epsilon_0 \\ \chi(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 v_0}{\partial x^2} \end{bmatrix} = \mathbb{B} q_e$$

where

Strain matrix expression

$$\mathbb{B} = \begin{bmatrix} N_1' & 0 & 0 & N_2' & 0 & 0 \\ 0 & H_1'' & H_2'' & 0 & H_3'' & H_4'' \end{bmatrix}$$

Moreover

$$u_b = \begin{bmatrix} u_0 \\ v_0 \\ \beta \end{bmatrix} = \mathbb{N} q_e \quad \text{and} \quad f_v = \begin{bmatrix} f_x^v \\ f_y^v \\ f_{Rot}^v = 0 \end{bmatrix}$$

ε_f matrix form demonstration :

$$\varepsilon_f = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 v_0}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{bmatrix}$$

where

$$\begin{cases} u_0(x) = N_1(x)\, u_1 + N_2(x)\, u_2 \\ v_0(x) = H_1(x)\, v_1 + H_2(x)\, \beta_1 + H_3(x)\, u_2 + H_4(x)\, \beta_2 \end{cases}$$

thus

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \underbrace{\begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & H_1 & H_2 & 0 & H_3 & H_4 \end{bmatrix}}_{N_{uv}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \beta_1 \\ u_2 \\ v_2 \\ \beta_2 \end{bmatrix}}_{q_e}$$

And

$$\varepsilon_f = \underbrace{\begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial^2/\partial x^2 \end{bmatrix} N_{uv}}_{\mathbb{B}}\, q_e = \underbrace{\begin{bmatrix} N_1' & 0 & 0 & N_2' & 0 & 0 \\ 0 & H_1'' & H_2'' & 0 & H_3'' & H_4'' \end{bmatrix}}_{\mathbb{B}} q_e$$

FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

Integration over the potato

Integrating on each FE, we have

$\int_I (\varepsilon_b^{\star})^T \sigma_b \, dx$

$$\sum_{EF} \int_{\Gamma_e} (\epsilon_b^{\star})^T \sigma_b dx = \sum_{EF} \left(\int_{\Gamma_e} (u_b^{\star})^T f_v \, dx + \sum_{\partial\Gamma_e} (u_b^{\star})^T F_{bs} \right)$$

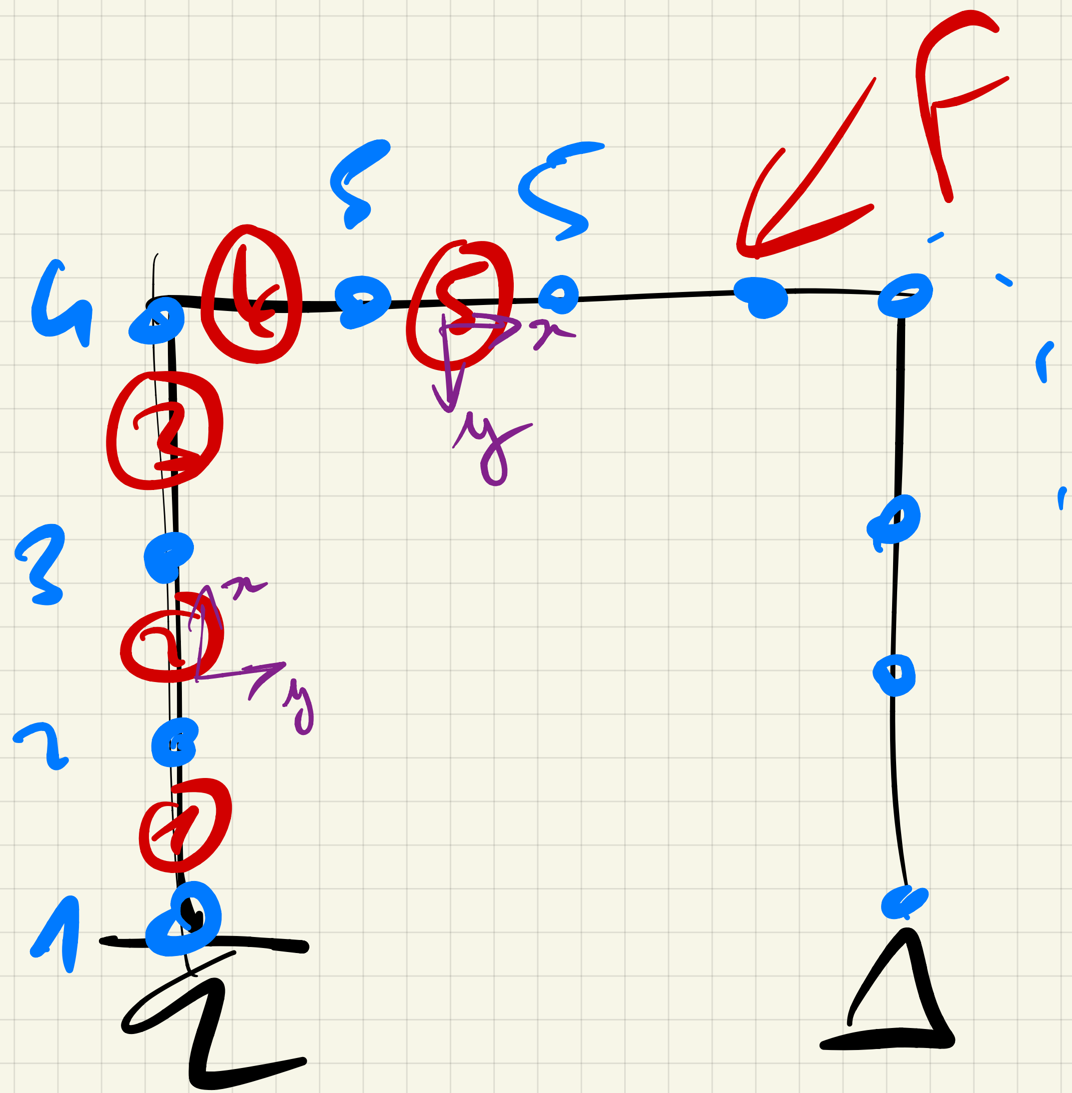
which is equivalent to

local frame

Matrix form

$$\sum_{EF} \int_0^{L_e} (q_e^{\star})^T \mathbb{B}^T \sigma_b dx = \sum_{EF} \left(\int_0^{L_e} (q_e^{\star})^T \mathbb{N}^T f_v \, dx + \sum_{\partial\Gamma_e} (q_e^{\star})^T \mathbb{N}^T F_{bs} \right)$$

EBFE



FEA in *quasi*-static conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Assembling step $\mathcal{A}_{EF}$

$$(U^\star)^T \mathcal{A}_{EF} \int_0^{L_e} \mathbb{B}^T \sigma_b dx = (U^\star)^T \mathcal{A}_{EF} \left(\int_0^{L_e} \mathbb{N}^T f_v \, dx + \sum_{\partial \Gamma_e} \mathbb{N}^T F_{bs} \right)$$

where U is the vector of nodal displacements in the global frame

> **Remark** : In local (resp. global) frame, lower-case (resp. upper-case) letters are used for the quantities (example: q_e (resp. Q_e))

FEA in *quasi*-static conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Classical mechanical balance expression (in the global frame)

$$\mathcal{A}_{EF} \int_0^{L_e} \mathbb{B}^T \sigma_b dx = \mathcal{A}_{EF} \left(\int_0^{L_e} \mathbb{N}^T f_v \, dx + \sum_{\partial \Gamma_e} \mathbb{N}^T F_{bs} \right)$$

Valid for linear and non linear materials

where we define

$$\begin{cases} F_{int}(U) = \mathcal{A}_{EF} \int_0^{L_e} \mathbb{B}^T \sigma_b dx \\ F_{ext} = \mathcal{A}_{EF} \left(\int_0^{L_e} \mathbb{N}^T f_v \, dx + \sum_{\partial \Gamma_e} \mathbb{N}^T F_{bs} \right) \end{cases}$$

FEA in *quasi*-static conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Resolution

→ We are elastic linear

Here, linear case thus $\sigma_b = \mathbb{C}\,\underbrace{\mathbb{B}q_e}_{\epsilon_b}$ **where C is constant**

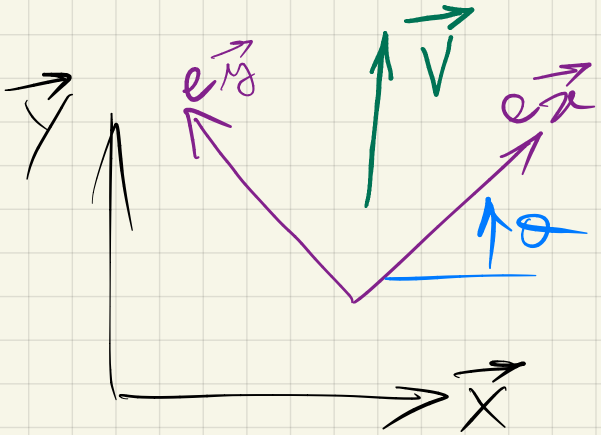
$$F_{int}(U) = \left[\mathcal{A}_{EF}\int_0^{L_e} \mathbb{B}^T\mathbb{C}\mathbb{B}dx\right]U \quad \textbf{thus} \quad \mathbb{K}U = F_{ext}$$

where $K_e = \int_0^{L_e} \mathbb{B}^T\mathbb{C}\mathbb{B}dx$ is the elementary stiffness matrix

$\mathbb{K} = \mathcal{A}_{EF}\int_0^{L_e} \mathbb{B}^T\mathbb{C}\mathbb{B}dx$ is assembled stiffness matrix

Global frame

Rotation matrix reminder (for a classic 2D vector $\vec{V}$)



$$\begin{cases} \vec{V}\Big|_{\vec{x}\vec{y}} = v_x \vec{e_x} + v_y \vec{e_y} \\ \vec{V}\Big|_{\vec{X}\vec{Y}} = V_X \vec{X} + V_Y \vec{Y} \end{cases} \quad \text{and} \quad \begin{cases} \vec{e_x} = \cos\theta\, \vec{X} + \sin\theta\, \vec{Y} \\ \vec{e_y} = -\sin\theta\, \vec{X} + \cos\theta\, \vec{Y} \end{cases}$$

$$\vec{V}\Big|_{\vec{x}\vec{y}} = v_x\left(\cos\theta\, \vec{X} + \sin\theta\, \vec{Y}\right) + v_y\left(-\sin\theta\, \vec{X} + \cos\theta\, \vec{Y}\right)$$

$$= \left(v_x \cos\theta - v_y \sin\theta\right)\vec{X} + \left(v_x \sin\theta + v_y \cos\theta\right)\vec{Y}$$

Thus:

$$\begin{bmatrix} V_X \\ V_Y \end{bmatrix} = \begin{bmatrix} v_x \cos\theta - v_y \sin\theta \\ v_x \sin\theta + v_y \cos\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} v_x \\ v_y \end{bmatrix}}_{\vec{V}|_{\vec{x}\vec{y}}}$$

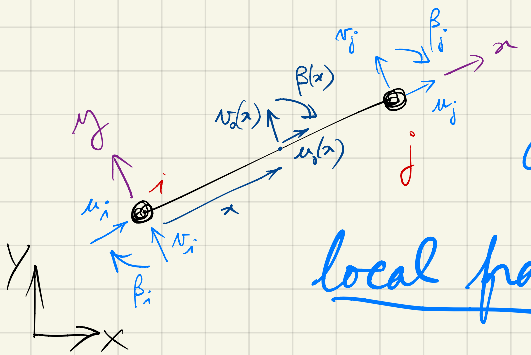
Finally

$$P = \left[\, \vec{e_x} \quad \vec{e_y} \,\right]$$

Passage matrix from local $(\vec{x},\vec{y})$ to global $(\vec{X},\vec{Y})$

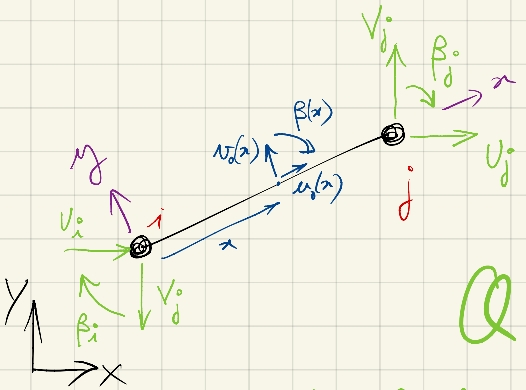
Within FEM (beam FE)

$$\underset{\text{(local)}}{q_e} \longrightarrow \underset{\text{(global)}}{Q_e} = P_6\, q_e \quad \text{where} \quad \begin{cases} P_6 = \left[\begin{array}{ccc|c} [P] & & \begin{matrix}0\\0\end{matrix} & 0_{3\times3} \\ 0 & 0 & 1 & \\ \hline & 0_{3\times3} & & \begin{matrix}[P] & \begin{matrix}0\\0\end{matrix} \\ 0\ \ 0 & 1\end{matrix} \end{array}\right] \\ P = \left[\vec{e}_x \ \ \vec{e}_y\right] \end{cases}$$



q_e

local frame



Q_e

global frame

$$\underbrace{\begin{bmatrix} U_1 \\ V_1 \\ \beta_1 \\ \hdashline U_2 \\ V_2 \\ \beta_2 \end{bmatrix}}_{Q_e} = \underbrace{\left[\begin{array}{ccc|c} [P] & & \begin{matrix}0\\0\end{matrix} & 0_{3\times3} \\ 0 & 0 & 1 & \\ \hline & 0_{3\times3} & & \begin{matrix}[P] & \begin{matrix}0\\0\end{matrix} \\ 0\ \ 0 & 1\end{matrix} \end{array}\right]}_{P_6} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \beta_1 \\ \hdashline u_2 \\ v_2 \\ \beta_2 \end{bmatrix}}_{q_e}$$

Thus, at the element scale within the local frame

$$f_e = k_e \, q_e$$

f_e → nodal forces

k_e → stiffness matrix of the FE

q_e → nodal displacements

It can be move to the global frame through

$$\begin{cases} Q_e = P_6 \, q_e \\ F_e = P_6 \, f_e \end{cases} \Rightarrow \quad P_6^T F_e = k_e \, P_6^T Q_e$$

Rq: $P_6^{-1} = P_6^T$

thus $$F_e = \underbrace{P_6 \, k_e \, P_6^T}_{K_e} \, Q_e$$

K_e : stiffness matrix of FE within the global frame

⟹ can be assembled

FEA in *quasi*-static conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Elementary stifness matrix of EBFE in the local frame

$$k_e = \begin{bmatrix} \frac{ES}{L_e} & 0 & 0 & -\frac{ES}{L_e} & 0 & 0 \\ 0 & \frac{12EI}{L_e^3} & \frac{6EI}{L_e^2} & 0 & -\frac{12EI}{L_e^3} & \frac{6EI}{L_e^2} \\ 0 & \frac{6EI}{L_e^2} & \frac{4EI}{L_e} & 0 & -\frac{6EI}{L_e^2} & \frac{2EI}{L_e} \\ -\frac{ES}{L_e} & 0 & 0 & \frac{ES}{L_e} & 0 & 0 \\ 0 & -\frac{12EI}{L_e^3} & -\frac{6EI}{L_e^2} & 0 & \frac{12EI}{L_e^3} & -\frac{6EI}{L_e^2} \\ 0 & \frac{6EI}{L_e^2} & \frac{2EI}{L_e} & 0 & -\frac{6EI}{L_e^2} & \frac{4EI}{L_e} \end{bmatrix} \quad [q_e] = \begin{bmatrix} u_1 \\ v_1 \\ \beta_1 \\ u_2 \\ v_2 \\ \beta_2 \end{bmatrix}$$

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Outline

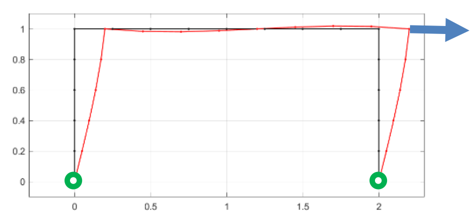
FEA in *quasi*-static conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

How to do it in MATLAB ?

D:\Enseignements\ModulesOptionnels\M8
\CM\Poly\Figures\QS**CantileverBeam**

D:\Enseignements\ModulesOptionnels\M8
\CM\Poly\Figures\QS**FrameStrExample**

Example of a 2D portal frame
Loading : ponctual load
Boundary conditions :
articulation the supports



INPUTS (Geometry, Material, etc.)

↓

FE data
(E, I, L_e, g, rho, *etc.*)

↓

Meshing
(XY, TC, DOF)

↓

Matrix Assembling
(M, K, C)

↓

Loading and Boundary conditions

↓

Resolution

↓

Post Processing

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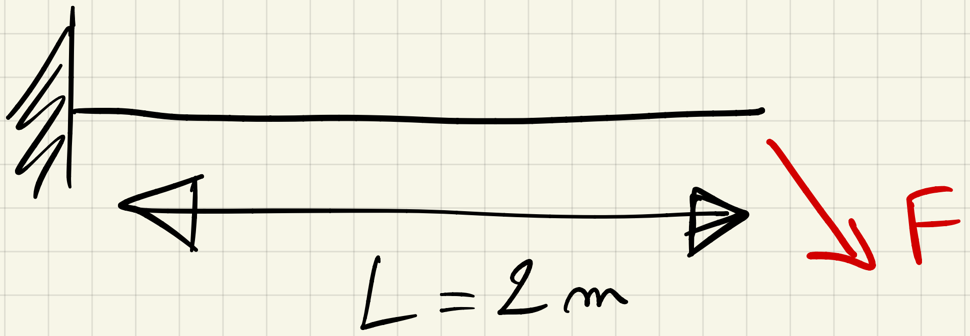
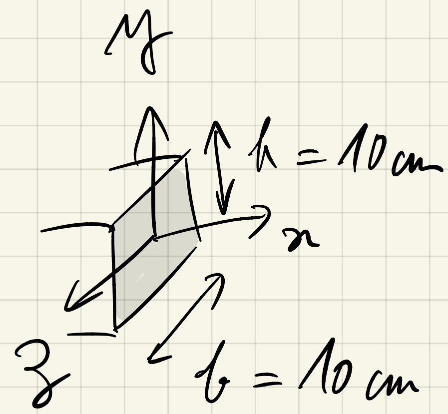
Before : Simples "By Hand" examples

Example ①:

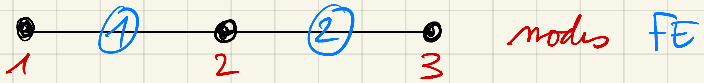
To do with MATLAB





→ $(x,y) = (X,Y)$ no rotation needed.
local   global

→ Mesh : 2 EBFE

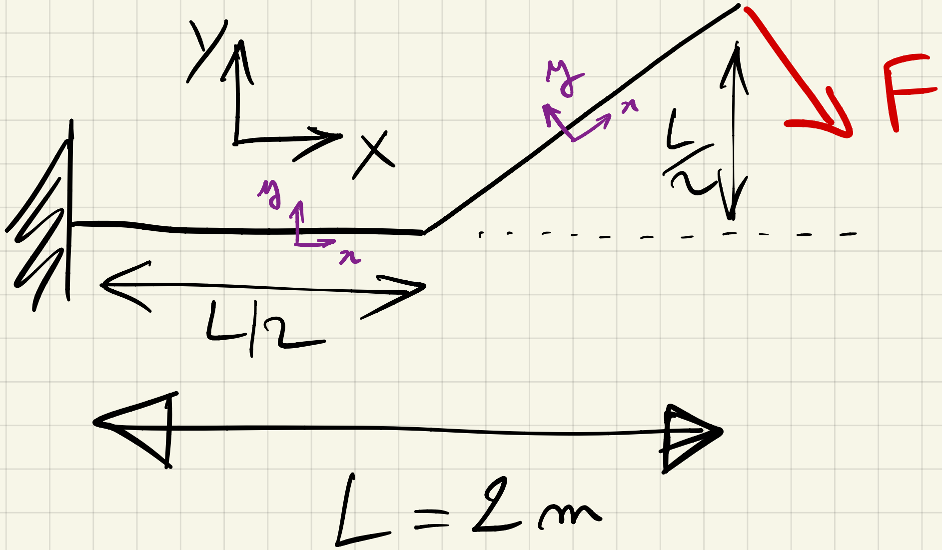


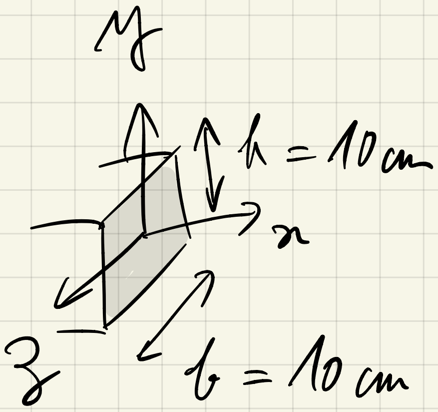
→ Make the assembling of the system $K\,U = F_{ext}$

→ Find U and plot the deformed shape.

Example ②:

To do with MATLAB

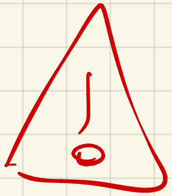




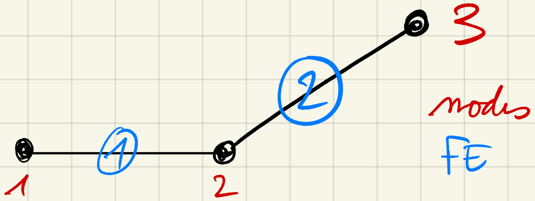
$Q_e = P_6 \, q_e$

Rq: $K_e = P_6 \, k_e \, P_6^T$

↓ stiffness matrix within local frame

→  $(x,y) \neq (X,Y)$ rotation needed.
local global

→ Mesh: 2 EBFE



→ Make the assembling of the system $\mathbb{K} \, U = F_{ext}$

→ Find U and plot the deformed shape.

Outline

1. Introduction

2. FEA in *quasi*-static conditions
   - 2.1. Theoretical aspects
   - 2.2. Algorithmic aspects

3. FEA in dynamic conditions
   - 3.1. Theoretical aspects
   - 3.2. Algorithmic aspects

4. Projets aims

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FEA in dynamic conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Principle of Virtual Work

$$\underbrace{\int_V (u^\star)^T \rho\, \ddot{u}\, dV}_{\text{Work of inertial forces}} + \int_\Gamma (\epsilon_b^\star)^T \sigma_b d\Gamma = \int_\Gamma (u_b^\star)^T f_v\, d\Gamma + \sum_{\partial\Gamma} (u_b^\star)^T F_{bs}$$

$$\text{where } \ddot{u} = \frac{\partial^2 u}{\partial t^2} \text{ is the acceleration of a given material point}$$

Remainder : $\begin{bmatrix} u^\star \\ v^\star \end{bmatrix} = \begin{bmatrix} u_0^\star - y\,\beta^\star \\ v_0^\star \end{bmatrix}$ and $\begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} = \begin{bmatrix} \ddot{u}_0 - y\,\ddot{\beta} \\ \ddot{v}_0 \end{bmatrix}$

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FEA in dynamic conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

Principle of Virtual Work
⇒ Work of inertial forces

$$\int_V (u^\star)^T \rho\, \ddot{u}\, dV = \sum_{EF} \left(\int_{V_e} (u^\star)^T \rho\, \ddot{u}\, dV \right)$$

$$u^\star = \begin{bmatrix} u^\star \\ v^\star \end{bmatrix}$$

FE scale

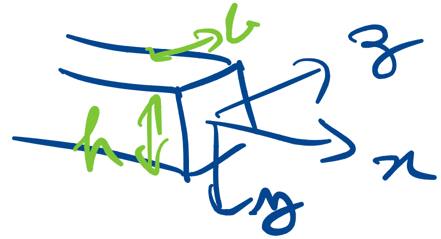
$$\int_{V_e} (u^\star)^T \rho\, \ddot{u}\, dV = \int_{V_e} \rho (u^\star \ddot{u} + v^\star \ddot{v})\, dV$$

$$= \int_{V_e} \rho \left[(u_0^\star - y\, \beta^\star)(\ddot{u}_0 - y\, \ddot{\beta}) + v_0^\star \ddot{v}_0 \right] dV$$

FEA in dynamic conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

Principle of Virtual Work
⇒ Work of inertial forces

$$\int_{V_e} (u^\star)^T \rho\, \ddot{u}\, dV = \rho \int_0^{L_e} \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} u_0^\star \ddot{u}_0 - y\left(u_0^\star \ddot{\beta} + \ddot{u}_0 \beta^\star\right) + y^2 \left(\beta^\star \ddot{\beta}\right) + v_0^\star \ddot{v}_0 \right) b\, dy\, dx$$



FE scale

$$\rho \int_0^{L_e} \left(S\, u_0^\star \ddot{u}_0 + S\, v_0^\star \ddot{v}_0 + I_{Gz}\, \beta^\star \ddot{\beta}\right) dx = \rho \int_0^{L_e} u_b^{\star T} \mathbb{C}_m\, \ddot{u}_b\, dx$$

$$\mathbb{C}_m = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I_{Gz} \end{bmatrix}$$

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FEA in dynamic conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Elementary mass matrix of EBFE (local frame)

$$u_b = \begin{bmatrix} u_0 \\ v_0 \\ \beta \end{bmatrix}$$

$$\int_{V_e} (u^\star)^T \rho\, \ddot{u}\, dV = \rho \int_0^{L_e} (q_e^\star)^T\, \mathbb{N}^T\, \mathbb{C}_m\, \mathbb{N}\, \ddot{q}_e\, dx$$

FE scale

Remainder : $u_b = \mathbb{N} q_e$ and $\ddot{u}_b = \mathbb{N} \ddot{q}_e$

$$m_e = \rho \int_0^{L_e} \mathbb{N}^T\, \mathbb{C}_m\, \mathbb{N}\, dx$$

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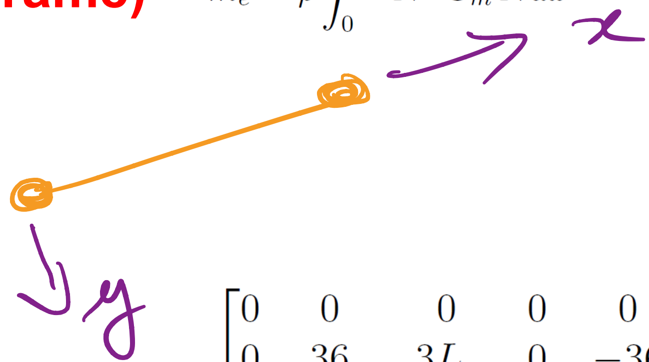
FEA in dynamic conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Elementary mass matrix of EBFE (local frame)

$$m_e = \rho \int_0^{L_e} \mathbb{N}^T \, \mathbb{C}_m \, \mathbb{N} \, dx$$



$$m_e = m_e^{\texttt{tra}} + m_e^{\texttt{rot}}$$

$$m_e^{\texttt{tra}} = \frac{\rho S L_e}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L_e & 0 & 54 & -13L_e \\ 0 & 22L_e & 4L_e^2 & 0 & 13L_e & -3L_e^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L_e & 0 & 156 & -22L_e \\ 0 & -13L_e & -3L_e^2 & 0 & -22L_e & 4L_e^2 \end{bmatrix}$$

Mass terms related to translations

$$m_e^{\texttt{rot}} = \frac{\rho \, I_{Gz}}{30 L_e} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3L_e & 0 & -36 & 3L_e \\ 0 & 3L_e & 4L_e^2 & 0 & -3L_e & -L_e^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3L_e & 0 & 36 & -3L_e \\ 0 & 3L_e & -L_e^2 & 0 & -3L_e & 4L_e^2 \end{bmatrix}$$

Mass terms related to rotations

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FEA in dynamic conditions
⇒ Formulation of Euler-Bernoulli beam FE in 2D

System formulation

Summation over the FE

$$\sum_{EF} (q_e^\star)^T \left(\int_0^{L_e} \rho \mathbb{N}^T \mathbb{C}_m \mathbb{N}\, dx\, \ddot{q}_e + \int_0^{L_e} \mathbb{B}^T \sigma_b dx \right) = \sum_{EF} (q_e^\star)^T \left(\int_0^{L_e} \mathbb{N}^T f_v\, dx + \sum_{\partial \Gamma_e} \mathbb{N}^T F_{bs} \right)$$

Assembling over the FE (global frame)

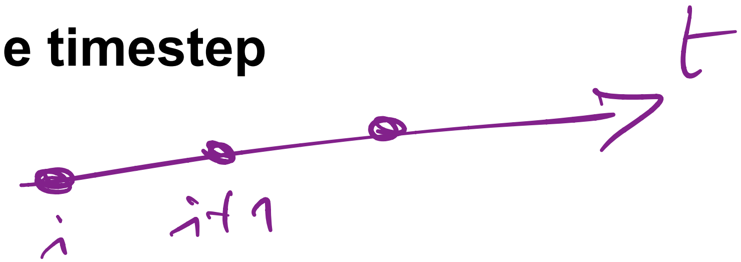
$$(U^\star)^T \left[\mathcal{A}_{EF} \left(\int_0^{L_e} \rho \mathbb{N}^T \mathbb{C}_m \mathbb{N}\, dx \right) \ddot{U} + \mathcal{A}_{EF} \left(\int_0^{L_e} \mathbb{B}^T \sigma_b dx \right) \right] = (U^\star)^T \mathcal{A}_{EF} \left(\int_0^{L_e} \mathbb{N}^T f_v\, dx + \sum_{\partial \Gamma_e} \mathbb{N}^T F_{bs} \right)$$

Final linear system

$$M\, \ddot{U}(t) + F_{int}(U(t)) = F_{ext}(t)$$

FEA in dynamic conditions
$\Rightarrow$ **Formulation of Euler-Bernoulli beam FE in 2D**

Resolution through time
$\Rightarrow$ **i is the time index : t_i = i dt where dt is the timestep**



Objective : find what's happen at i+1

Equation to solve $P(U_{i+1}, \dot{U}_{i+1}, \ddot{U}_{i+1}) = F_{ext}^{i+1}$

where $P(U_{i+1}, \dot{U}_{i+1}, \ddot{U}_{i+1}) = M\ddot{U}_{i+1} + C\dot{U}_{i+1} + F_{int}(U_{i+1})$

Use of Taylor expansions

FEA in dynamic conditions

⇒ **Formulation of Euler-Bernoulli beam FE in 2D**

Resolution through time

⇒ **Newmark integration scheme familly** allow to get velocities and accelerations at the next step all expressed as a function of U_{i+1}

$\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$ →

$$\begin{cases} \dot{U}_{i+1} = \dot{U}_i + (1-\gamma)\Delta t \ddot{U}_i + \dfrac{\gamma}{\beta \Delta t}\left[\boxed{U_{i+1}} - U_i - \Delta t \dot{U}_i - (\frac{1}{2} - \beta)\Delta t^2 \ddot{U}_i\right] \\ \ddot{U}_{i+1} = \dfrac{1}{\beta \Delta t^2}\left[\boxed{U_{i+1}} - U_i - \Delta t \dot{U}_i - (\frac{1}{2} - \beta)\Delta t^2 \ddot{U}_i\right] \end{cases}$$

⇒ **Displacement formulation**

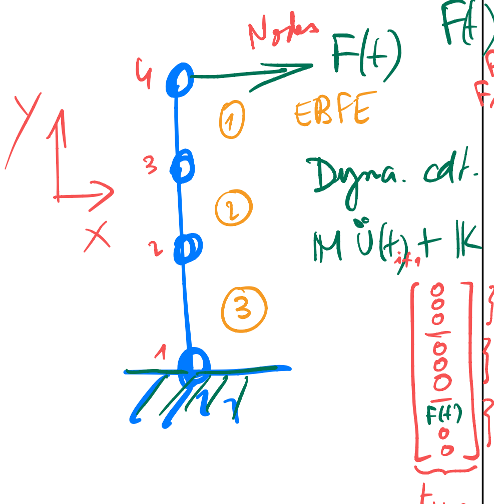
→ MS on moodle to get more info.

FEA in dynamic conditions

⇒ Formulation of Euler-Bernoulli beam FE in 2D

Resolution through time

⇒ Linear system to solve at each timestep



For **linear** system, we have $F_{int}(U_{i+1}) = KU_{i+1}$ which leads to

$$\underbrace{\left(K + \frac{\gamma}{\beta\Delta t}C + \frac{1}{\beta\Delta t^2}M\right)}_{S_{pf}} U_{i+1} =$$

$$f_{i+1}\left\{ F_{ext}^{i+1} + \left(\frac{\gamma}{\beta\Delta t}C + \frac{1}{\beta\Delta t^2}M\right)U_i + \left(C(\frac{\gamma}{\beta}-1) + \frac{1}{\beta\Delta t}M\right)\dot{U}_i + \left(C\Delta t\left(\frac{\gamma}{\beta}(\frac{1}{2}-\beta)-(1-\gamma)\right) + \frac{\frac{1}{2}-\beta}{\beta}M\right)\ddot{U}_i \right. \quad (3.23)$$

or in matrix form

$$S_{pf}U_{i+1} = f_{i+1} \quad (3.24)$$

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Outline

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FEA in dynamic conditions

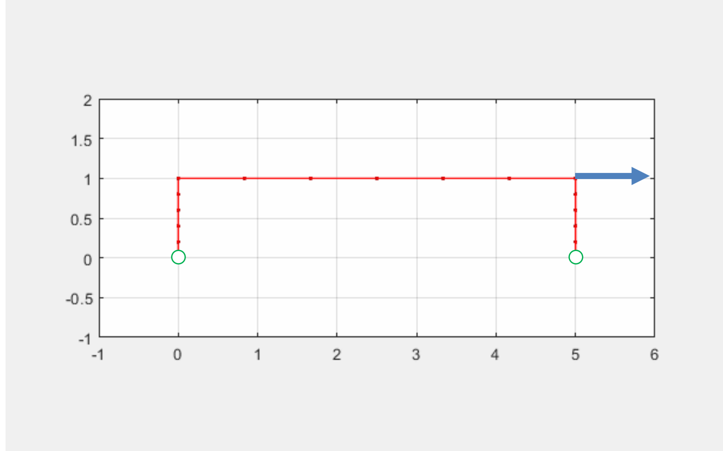
⇒ Formulation of Euler-Bernoulli beam FE in 2D

How to do it in MATLAB ?

D:\Enseignements\ModulesOptionnels\M8\CM\Poly\Figures\Dyna**CantileverBeam**

D:\Enseignements\ModulesOptionnels\M8\CM\Poly\Figures\Dyna**FrameStrExample**

Example of a 2D portal frame
Loading : ponctual load – **Heavyside through time**
Boundary conditions : articulation the supports



INPUTS (Geometry, Material, etc.)

↓

FE data
(E, I, L_e, g, rho, *etc.*)

↓

Meshing
(XY, TC, DOF)

↓

Matrix Assembling
(M, K, C)

↓

Loading and Boundary conditions

↓

Resolution

↓

Post Processing

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Outline

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Projets aims

Global project scale

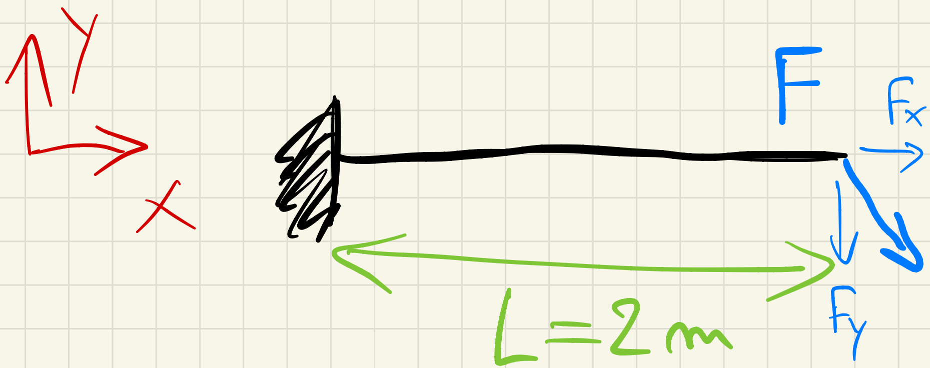
- Design a building using the fundamental principles of structural design
- Propose 1/200 scale plans
- Develop the ability to implement calculation methods in algorithms
- Optimize solutions according to multiple criteria
- Collaborate to solve complex problems

Structure scale

- Design and sizing of a post-beam structure
- Development of the FE model in linear elasticity (2D) from *Euler-Bernoulli* type beam elements (under MATLAB).
- Maximum internal forces and displacements assessment and cross sections sizing according to the material considered
- Analysis of the effect of different loading scenarii (dead weight, wind, operating load, *etc.*) on sizing

Quasi static cdt → $\mathbb{K} U = \mathbb{F}$

Cantilever beam



Inputs

Linear elastic mtx

$E = 200e9;$

$b = 0,1;$

$h = 0,1;$

$\rho = 7500;$

F = [Fcdtx ; Fcdty];

Fcdtx = 0; Fcdty = -100000;

FE → EBFE

mEFpoint = 10;