FEM in linear elasticity
and application for shallow foundation

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Plan

Aim of the course

warning

This course is intended to provide the necessary theoretical elements to be able to write a first FEM code with Matlab®. Many documents exists about this topic in university libraries and on internet. As a consequence, readers and attendees have to find some of missing information according to the physical problem they have to deal with by themselves).

Continuous approach

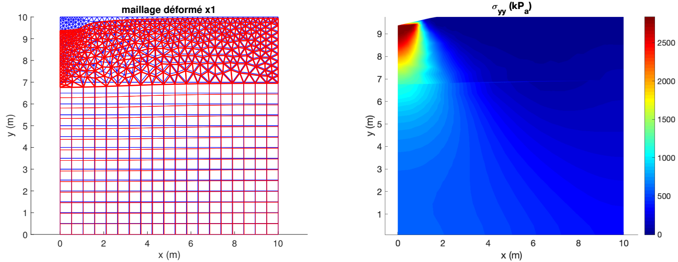


Figure – Example of results coming from FEM computation.

Introductory exercise I

We consider the structure displayed on Fig.2. It is constituted of an assembly of bars working in tension-compression only. They are linked by pure spherical joints.

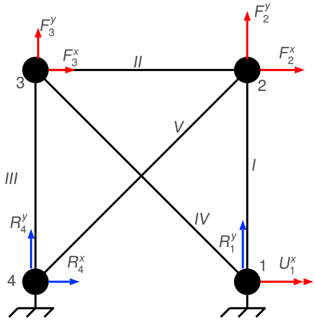


Figure – Mechanical linkage of studied structure

Introductory exercise II

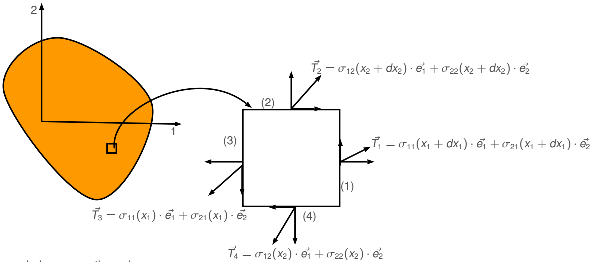
1. Boundary conditions : node 1 is blocked in y direction, node 4 is blocked in x and y direction.
2. Write the balance equations system at each node for the structure under the form $\{F_{int}\} = \{F_{ext}\}$. Prove that it is hyper static.
3. Introduce the constitutive relationship in the term $\{F_{int}\}$ in order to rewrite it as a function of the nodal displacements. We assume a pure linear elastic behavior for each bar. The rigidity is denoted k_i.
4. Rewrite the system by taking into account the boundary conditions.
5. Solve this system using Matlab ®.

P_1	P_2	P_3	P_4
$(1;0)$	$(1;1)$	$(0;1)$	$(0;0)$

k_I	k_{II}	k_{III}	k_{IV}	k_V
$20.e3$	$16.e3$	$20.e3$	$7.e3$	$7.e3$
F_2^x	F_2^y	F_3^x	F_3^y	U_1^x
50	-140	20	-170	0.01

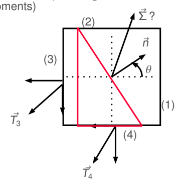
6. Using the virtual works principle, prove that this system can be built using an assembly of elementary stiffness matrices.

Cauchy stress tensor : "intuitive" building



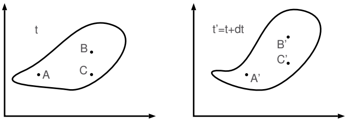
balance equations give :

1) $\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$ et $\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$ (forces)
2) $\sigma_{12} = \sigma_{21}$ (moments)



deduce the existence of a matrice $\underline{\underline{\sigma}}$ such that $\vec{\Sigma} = \underline{\underline{\sigma}} \cdot \vec{n}$.

Strain tensor I



$\left(\|\vec{A'B'}\|^2 - \|\vec{AB}\|^2\right)$ et $\left|\vec{A'B'} \cdot \vec{A'C'} - \vec{AB} \cdot \vec{AC}\right|$?

Consider the vectorial function $\vec{\phi}$ denoted *placing* function such that :

$$\vec{OA'} = \vec{\phi}(A, t)$$

thus, we get for any point B in the vicinity of A :

$$\vec{OB'} = \vec{\phi}(B) \underset{A}{=} \vec{\phi}(A) + \nabla\vec{\phi}(A) \cdot (\vec{OB} - \vec{OA}) + o(\vec{OB} - \vec{OA})$$

Thus

$$\begin{cases} \vec{A'B'} & \underset{A}{=} \nabla\vec{\phi}(A) \cdot \vec{AB} + o(\vec{AB}) \\ \vec{A'B'} & \approx \nabla\vec{\phi}(A) \cdot \vec{AB} = \underline{\underline{F}} \cdot \vec{AB} \end{cases}$$

Strain tensor II

and we get :

$$\begin{aligned} ||\vec{A'B'}||^2 - ||\vec{AB}||^2 &= \left(\underline{\underline{F}} \cdot \vec{AB}\right)^t \cdot \left(\underline{\underline{F}} \cdot \vec{AB}\right) - \vec{AB}^t \cdot \vec{AB} \\ &= \vec{AB}^t \underline{\underline{F}}^t \underline{\underline{F}} \vec{AB} - \vec{AB}^t \cdot \vec{AB} \\ &= \vec{AB}^t \left(\underline{\underline{F}}^t \underline{\underline{F}} - \underline{\underline{I}}\right) \vec{AB} \end{aligned}$$

and

$$\begin{aligned} \left|\vec{A'B'} \cdot \vec{A'C'} - \vec{AB} \cdot \vec{AC}\right| &= \left(\underline{\underline{F}} \cdot \vec{AB}\right)^t \cdot \left(\underline{\underline{F}} \cdot \vec{AC}\right) - \vec{AB}^t \cdot \vec{AC} \\ &= \vec{AB}^t \underline{\underline{F}}^t \underline{\underline{F}} \vec{AC} - \vec{AB}^t \cdot \vec{AC} \\ &= \vec{AB}^t \left(\underline{\underline{F}}^t \underline{\underline{F}} - \underline{\underline{I}}\right) \vec{AC} \end{aligned}$$

The tensor $\boxed{\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{F}}^t \underline{\underline{F}} - \underline{\underline{I}}\right)}$ is named *Green-Lagrange* tensor.

Strain tensor III

We can introduce the displacement vector :

$$\vec{u} = \vec{AA'} = \vec{OA'} - \vec{OA} \Leftrightarrow \vec{OA'} = \vec{OA} + \vec{u}$$

$$\underset{\text{by differentiation}}{\Rightarrow} \quad \nabla\vec{\phi} = \underline{\underline{I}} + \nabla\vec{u}$$

and

$$\boxed{\underline{\underline{E}} = \frac{1}{2}\left(\nabla\vec{u} + \nabla^t\vec{u} + \nabla^t\vec{u}\cdot\nabla\vec{u}\right)}$$

By neglecting the second order terms, we get the linearized strain tensor (small strains) :

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla u + \nabla^t u\right)}$$

notations about tensors

In this document Einstein's notations are used. Some basic reminders about these notations :

- repeated index : $a_i b_i = \sum_{i=1}^{3} a_i b_i$ ou encore $a_{ii} = \sum_{i=1}^{3} a_{ii}$
- spatial partial differentiation : $a_{,j} = \dfrac{\partial a}{\partial x_j}$
- time differentiation : basic *dot* notation : $\dot{a} = \dfrac{\partial a}{\partial t}$
- Kronecker's symbol : $\delta_{ij} = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{else} \end{cases}$
- permutation pseudo tensor : $\epsilon_{ijk} = \begin{cases} 0 & \text{si i=j ou j=k ou k=i} \\ 1 & \text{si i,j,k in the direct way} \\ -1 & \text{si i,j,k in the opposite way} \end{cases}$

reminders on matrices of base permutation I

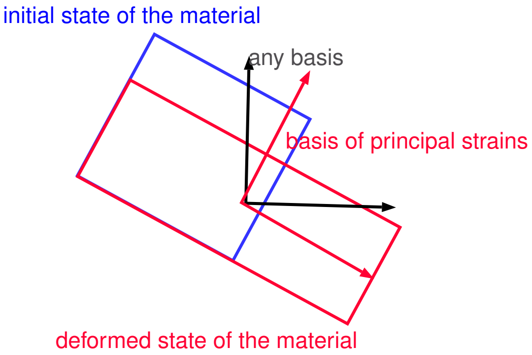


Figure – motivation

reminders on matrices of base permutation II

Théorème

If X_0 and X_1 are the colon vectors of the coordinates of a same vector u expressed in the bases B_0 and B_1 of the $\mathbb{K} - ev\ \mathcal{E}$, we get :

$$X_0 = P_{B_0 \to B_1} X_1$$

with $P_{B_0 \to B_1}$ the matrix of the base vector collection of the base B_1 expressed in the base B_0, and sorted in column.

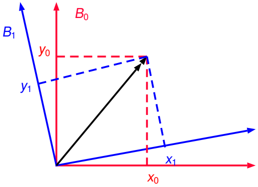


Figure – effect of moving base on a vector

reminders on matrices of base permutation III

effect of moving base on a matrice

Let $\mathcal{E}$ and $\mathcal{F}$ being two $\mathbb{K}-ev$, B_{E0} and B_{E1} two bases of $\mathcal{E}$ of B_{F0} and B_{F1} two bases of $\mathcal{F}$. We denote P the moving matrice form B_{E0} to B_{E1}, and by Q the one of B_{F0} to B_{F1}. Let be K_0 the matrice of a linear application from $\mathcal{E}$ in $\mathcal{F}$ written in the bases B_{E0}, B_{F0} (such that $F_0 = K_0 X_0$).

Show that K_1 the matrice of this same linear application written in the bases B_{E1}, B_{F1} is computed as follow :

$$K_1 = Q^{-1} K_0 P$$

Reminders on the reduction of symmetric matrices

Théorème

Let f be a real symmetric endomorphisme ($f : E \rightarrow E$) :

- *f is diagonalisable on $\mathbb{R}$*
- *every eigenvalues are real*
- *its eigen subspaces are 2 to 2 orthogonal*

ie : f diagonalizes into $\mathbb{R}$ on an orthonormal basis.

Théorème

Let M be a real symmetrical square matrix. There is :

- *D a diagonal matrix with real coefficients*
- *P an orthogonal matrix (i.e. orthonormal basis change)*

such that : $M = P \cdot D \cdot P^{-1} = P \cdot D \cdot P^{t}$

Scalar

fundamental property on tensors

A tensor of order n is a mathematical object (usually describing physical quantities) that is written in different bases using a formula of basis change. example : $X = P \cdot \tilde{X}$, $\tilde{M} = P^{-1} \cdot M \cdot P.\tilde{X}$ and X represent the same tensor of order 1 and $\tilde{M}$ and M the same tensor of order 2. But the values of their components are different when they are expressed in different bases.

Définition

A scalar is a 0 order tensor. Therefore, a scalar is invariant by a basis change.
Examples include temperature, pressure, relative humidity...
versus, a component of a vector or a tensor.

notion of invariants of a matrix of rank 3

A *invariant* is a scalar function of a tensor which does not depend on the base considered. ⚠ Invariants are not unique for a given matrix, but families of invariants exist. The most used are :

- The one of Cayley-Hamilton (coefficients of the characteristic polynomial) $\det(\mathbf{M} - \lambda \mathbf{I}) = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3$
  - $\hat{I}_1 = \text{trace}(\mathbf{M})$
  - $\hat{I}_2 = \frac{1}{2}\left(\text{trace}(\mathbf{M})^2 - \text{trace}(\mathbf{M} \cdot \mathbf{M})\right)$
  - $\hat{I}_3 = \det(\mathbf{M})$
- Rivlin-Ericksen decomposition
  - $\tilde{I}_1 = \text{trace}(\mathbf{M})$
  - $\tilde{I}_2 = \frac{1}{2}\text{trace}(\mathbf{M} \cdot \mathbf{M})$
  - $\tilde{I}_3 = \frac{1}{3}\text{trace}(\mathbf{M} \cdot \mathbf{M} \cdot \mathbf{M})$

physical meaning of the first invariant

- expression :
$$\begin{cases} I_{1\sigma} = \text{tr}(\boldsymbol{\sigma}) = \sigma_{kk} = 3 \cdot p \text{ (pressure or mean stress)} \\ I_{1\varepsilon} = \text{tr}(\varepsilon) = \varepsilon_{kk} = \dfrac{\Delta V}{V_0} \text{ (in small strains)} \end{cases}$$
- graphical illustration :

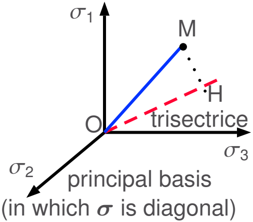


Figure – graphical illustration fo the stress tensor in its principal basis

- show that $OH = \frac{I_{1\sigma}}{\sqrt{3}} = \sqrt{3}p$

physical meaning of the du second invariant I

- deviatoric tensor :

$$\begin{cases} s_{ij} = \sigma_{ij} - \dfrac{I_{1\sigma}}{3}\delta_{ij} \\ e_{ij} = \varepsilon_{ij} - \dfrac{I_{1\varepsilon}}{3}\delta_{ij} \end{cases}$$

remark : $\boxed{s_{kk} = e_{kk} = 0}$

these tensors represent only the *shear* parts of the stress and strain tensors respectively.

- expression :

$$\begin{cases} J_{2s} = \mathrm{tr}(\boldsymbol{s}^2) \overset{(*)}{=} s_{ij}s_{ij} & (*) \text{ uniquement car } s_{ij} = s_{ji} \\ J_{2e} = \mathrm{tr}(\boldsymbol{e}^2) \overset{(*)}{=} e_{ij}e_{ij} \end{cases}$$

physical meaning of the du second invariant II

- graphical illustration :

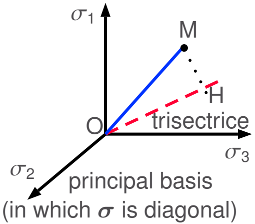


FIGURE – graphical illustration fo the stress tensor in its principal basis

- show that $\sqrt{J_{2s}} = HM$

physical meaning of the third invariant I

Définition

We call the deviatoric plan the ⊥ plan to the trisector passing through M considered in the space of the principal stresses.

Définition

The angle formed by the projection in the deviatoric plane of the axis 1 and the vector $\overrightarrow{HM}$ is called Lode's angle.

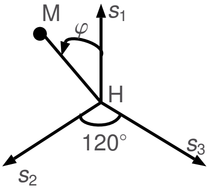


FIGURE – graphical illustration of the Lode's angle

physical meaning of the third invariant II

- expression :

$$\begin{cases} J_{3s} = \mathrm{tr}(\boldsymbol{s}^3) \overset{(*)}{=} s_{ij} s_{jk} s_{ki} \\ J_{3e} = \mathrm{tr}(\boldsymbol{e}^3) \overset{(*)}{=} e_{ij} e_{jk} e_{kj} \end{cases}$$

- we can show that :

$$\boxed{\cos(3\varphi_\sigma) = \frac{\sqrt{6} J_{3s}}{\sqrt{(J_{2s})^3}} \qquad \cos(3\varphi_\varepsilon) = \frac{\sqrt{6} J_{3e}}{\sqrt{(J_{2e})^3}}}$$

<u>conclusion</u> : $z = OH$, $r = MH$ and φ designate the cylindrical coordinates of the tensor expressed in its principal basis, and the trisector is the spinning axis.

démonstration I

(rien que pour vous : je ne l'ai trouvée nul part sur le net !!!)

1. préliminaire : opérateur de projection
   Soit Π le sous espace sur lequel on projette, $\vec{n}$ le vecteur unitaire indiquant la direction dans laquelle on projette, $\vec{v}$ le vecteur à projeter, et $\vec{v_p}$ le vecteur projeté.

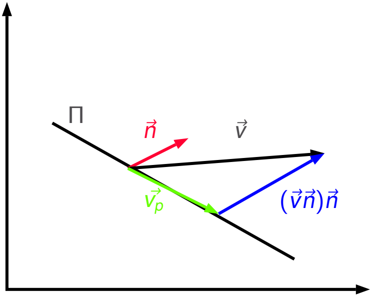


Figure – croquis projection

démonstration II

Le vecteur $\vec{v_p}$ s'écrit donc :

$$\vec{v_p} = \vec{v} - (\vec{v}\vec{n})\vec{n} = \vec{v} - (\vec{n} \otimes \vec{n})\vec{v} = \left(\underline{\underline{I}} - \vec{n} \otimes \vec{n}\right)\vec{v}$$

De là on tire la matrice de projection. Dans le cas particulier où :

$$\vec{n} = \begin{vmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{vmatrix}$$

, on trouve :

$$P = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

démonstration III

2. angle de Lode
Soit $\vec{\pi_1}$ le projeté de $\vec{e_1}$ dans le plan de Rendulic, On a :

$$\vec{\pi_1} = P\vec{e_1} = \frac{1}{3}\begin{vmatrix} 2 \\ -1 \\ -1 \end{vmatrix}$$

et $\vec{\pi_{1n}}$ le projeté normé :

$$\vec{\pi_{1n}} = \frac{1}{\sqrt{6}}\begin{vmatrix} 2 \\ -1 \\ -1 \end{vmatrix}$$

Donc :

$$\vec{\pi_{1n}}\vec{HM} = \left\|\vec{HM}\right\| \cos\varphi$$

et

$$\cos\varphi = \frac{\vec{\pi_{1n}}\vec{HM}}{\left\|\vec{HM}\right\|} = \frac{1}{\sqrt{6}}\frac{2S_1 - S_2 - S_3}{\sqrt{J_2}}$$

démonstration IV

or par construction du tenseur déviatorique, on a :

$$S_1 + S_2 + S_3 = 0 \Rightarrow 2S_1 - S_2 - S_3 = 3S_1$$

donc

$$\cos\varphi = \frac{\vec{\pi_{1n}}\vec{HM}}{\left\|\vec{HM}\right\|} = \frac{1}{\sqrt{6}}\frac{S_1}{\sqrt{J_2}}$$

en posant $\beta = \varphi + 120°$ et $\gamma = \varphi + 240°$ on a :

$$\cos\beta = \frac{1}{\sqrt{6}}\frac{S_3}{\sqrt{J_2}} \text{ et } \cos\gamma = \frac{1}{\sqrt{6}}\frac{S_2}{\sqrt{J_2}}$$

en utilisant :

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

on a :

$$\begin{aligned}&\cos^3(\varphi) + \cos^3(\beta) + \cos^3(\gamma) = \\ &\cos^3(\varphi) + \frac{1}{8}\left(\sqrt{(3)}\sin\varphi - \cos\varphi\right)^3 - \frac{1}{8}\left(\sqrt{(3)}\sin\varphi + \cos\varphi\right)^3 = \ldots \\ &= \frac{3}{4}\left(4\cos^3(\varphi) - 3\cos\varphi\right)\end{aligned}$$

démonstration V

or la linéarisation de $\cos^3 x$ donne :

$$\cos(3x) = 4\cos^3 x - 3\cos x$$

d'où

$$\cos^3(\varphi) + \cos^3(\beta) + \cos^3(\gamma) = \frac{3}{4}\cos(3\varphi)$$

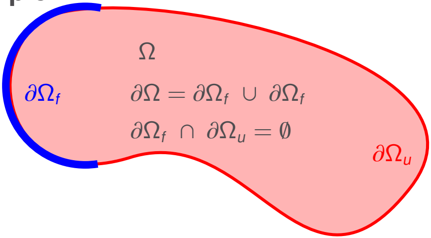
d'autre part,

$$\cos^3(\varphi) + \cos^3(\beta) + \cos^3(\gamma) = \frac{27}{6\sqrt{6}\sqrt{J_2^3}}\left(S_1^3 + S_2^3 + S_3^3\right) = \frac{9}{2\sqrt{6}}\frac{J_3}{\sqrt{J_2^3}}$$

d'où le résultat attendu !

$$\boxed{\cos(3\varphi) = \sqrt{6}\frac{J_3}{\sqrt{J_2^3}}}$$

Virtual work principle



$$\forall u^* \; CA, \underbrace{\int_\Omega \varepsilon^* : \boldsymbol{\sigma} d\Omega}_{T^*_{int}} = \underbrace{\int_\Omega f \cdot u^* d\Omega + \int_{\partial\Omega_f} t \cdot u^* dS}_{T^*_{ext} = T^*_{fv} + T^*_{fs}} \tag{1}$$

with

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

and

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{3}$$

Shape of C^{-1}_{ijkl} I

Isotropic case (true in any basis) :

$$\begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ \sqrt{2}d\varepsilon_{23} \\ \sqrt{2}d\varepsilon_{31} \\ \sqrt{2}d\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E^t} & -\frac{\nu^t}{E^t} & -\frac{\nu^t}{E^t} & 0 & 0 & 0 \\ -\frac{\nu^t}{E^t} & \frac{1}{E^t} & -\frac{\nu^t}{E^t} & 0 & 0 & 0 \\ -\frac{\nu^t}{E^t} & -\frac{\nu^t}{E^t} & \frac{1}{E^t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+\nu^t}{E^t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\nu^t}{E^t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu^t}{E^t} \end{bmatrix} \begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ \sqrt{2}d\sigma_{23} \\ \sqrt{2}d\sigma_{31} \\ \sqrt{2}d\sigma_{12} \end{bmatrix}$$

Orthotropic case in the principal axes (basis change to do if not) :

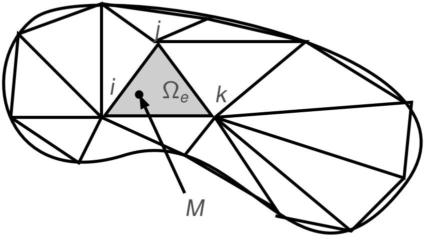
$$\begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ \sqrt{2}d\varepsilon_{23} \\ \sqrt{2}d\varepsilon_{31} \\ \sqrt{2}d\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E^t_1} & -\frac{\nu^t_{12}}{E^t_2} & -\frac{\nu^t_{13}}{E^t_3} & 0 & 0 & 0 \\ -\frac{\nu^t_{21}}{E^t_1} & \frac{1}{E^t_2} & -\frac{\nu^t_{23}}{E^t_3} & 0 & 0 & 0 \\ -\frac{\nu^t_{31}}{E^t_1} & -\frac{\nu^t_{32}}{E^t_2} & \frac{1}{E^t_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G^t_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G^t_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G^t_3} \end{bmatrix} \begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ \sqrt{2}d\sigma_{23} \\ \sqrt{2}d\sigma_{31} \\ \sqrt{2}d\sigma_{12} \end{bmatrix}$$

Shape of C^{-1}_{ijkl} II

Transverse isotropy case in the principal axes (basis change to do if not) :

$$\begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ \sqrt{2}d\varepsilon_{23} \\ \sqrt{2}d\varepsilon_{31} \\ \sqrt{2}d\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E^t_1} & -\frac{\nu^t_{12}}{E^t_2} & -\frac{\nu^t_{12}}{E^t_2} & 0 & 0 & 0 \\ -\frac{\nu^t_{21}}{E^t_1} & \frac{1}{E^t_2} & -\frac{\nu^t_{22}}{E^t_2} & 0 & 0 & 0 \\ -\frac{\nu^t_{21}}{E^t_1} & -\frac{\nu^t_{22}}{E^t_2} & \frac{1}{E^t_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+\nu^t_{22}}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G^t_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G^t_2} \end{bmatrix} \begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ \sqrt{2}d\sigma_{23} \\ \sqrt{2}d\sigma_{31} \\ \sqrt{2}d\sigma_{12} \end{bmatrix}$$

FEM discretization I



$$u(M) = \sum_{i=1}^{N_{nod}} u_i N_i(M) \tag{4}$$

By choosing convenient shape functions N.

$$u(M) = \sum_{i=1}^{N_{nod}^{el}} u_i^{el} N_i^{el}(M) = N_e u_e \tag{5}$$

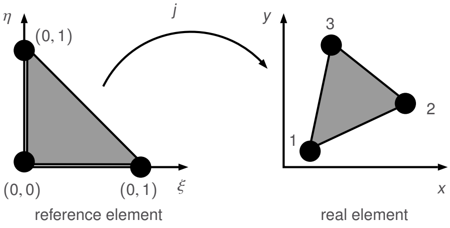
FEM discretization II

In order to respect the mass conservation, and the continuity of the displacement, the stress and the strain fields, the shape functions have to fulfilled the followings requirements :

$$\begin{cases} N_i\left(x_i\right) = 1 \\ N_i\left(x_j\right) = 0 \ \forall i \neq j \end{cases} \tag{6}$$

$$N_i \in \mathcal{C}^1 \text{ at least (depends on the physical problem to solve)} \tag{7}$$

Example of shape functions I



$$\left\{\begin{array}{l} N_1 = (1-\xi-\eta) \\ N_2 = \xi \\ N_3 = \eta \end{array}\right. \tag{8}$$

Remark :

$$\left\{\begin{array}{l} X(\xi,\eta) = N_k(\xi,\eta)\, X_k \\ Y(\xi,\eta) = N_k(\xi,\eta)\, Y_k \end{array}\right. \tag{9}$$

Example of shape functions II

$$J = \begin{bmatrix} N^t_{,\xi} \\ N^t_{,\eta} \end{bmatrix} \begin{bmatrix} X_{nod} & Y_{nod} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} X_1 + \frac{\partial N_2}{\partial \xi} X_2 + \frac{\partial N_3}{\partial \xi} X_3 & \frac{\partial N_1}{\partial \xi} Y_1 + \frac{\partial N_2}{\partial \xi} Y_2 + \frac{\partial N_3}{\partial \xi} Y_3 \\ \frac{\partial N_1}{\partial \eta} X_1 + \frac{\partial N_2}{\partial \eta} X_2 + \frac{\partial N_3}{\partial \eta} X_3 & \frac{\partial N_1}{\partial \eta} Y_1 + \frac{\partial N_2}{\partial \eta} Y_2 + \frac{\partial N_3}{\partial \eta} Y_3 \end{bmatrix} \tag{10}$$

$$j = J^{-1} \tag{11}$$

strain and stress field discretization

example for 2D problems :

$$\varepsilon = \begin{bmatrix} \dots & N^e_{i,x} & 0 & \dots \\ \dots & 0 & N^e_{i,y} & \dots \\ \dots & N^e_{i,y} & N^e_{i,x} & \dots \end{bmatrix} \begin{bmatrix} \dots \\ U^e_{ix} \\ U^e_{iy} \\ \dots \end{bmatrix} = B_e U_e \tag{12}$$

$$\varepsilon = \begin{bmatrix} \dots & j_{11} N^e_{i,\xi} + j_{12} N^e_{i,\eta} & 0 & \dots \\ \dots & 0 & j_{21} N^e_{i,\xi} + j_{22} N^e_{i,\eta} & \dots \\ \dots & j_{21} N^e_{i,\xi} + j_{22} N^e_{i,\eta} & j_{11} N^e_{i,\xi} + j_{12} N^e_{i,\eta} & \dots \end{bmatrix} \begin{bmatrix} \dots \\ U^e_{ix} \\ U^e_{iy} \\ \dots \end{bmatrix} \tag{13}$$

$$\boldsymbol{\sigma} = C_e B_e U_e \tag{14}$$

discretized virtual work principle

$$\sum_{nel}\left(U_e^{*t}\underbrace{\int_\Omega B_e^t C_e B_e d\Omega}_{K_e} U_e\right) = \sum_{nel}\left[U_e^{*t}\underbrace{\int_{\Omega_e} N_e^t f d\Omega}_{f_{ve}}\right] + \sum_{el}\left[U_e^{*t}\underbrace{\int_{\partial\Omega_e} N_e^t t dS}_{f_{se}}\right] \quad (15)$$

U_e^* and U_e are independent of Ω and $\partial\Omega$. Furthermore, the principle is true whatever the virtual displacement. As a consequence, a linear problem ca be built in the same manner than already seen in the first exercise. It is also possible to build every sub-system for each element and then assemble them in the global system.

$$\underbrace{KU}_{\vec{f}_{int}} = \underbrace{F}_{\vec{f}_{ext}} \quad (16)$$

evaluation of K_e

$$K_e = \int_{\Omega_e} B^t C B d\Omega = \int_{\Omega_{ref}} B^t C B \left|\det(J)\right| d\xi d\eta \tag{17}$$

Most of the time, this integration is not trivial and is performed numerically using a Gauss method. In this method, the different quantity to be integrated are evaluated at some given *integration points* and added with a weighting ratio.

$$K_e = \int_{\Omega_e} B^t C B d\Omega = \sum_{npi} w_i \left(B^t C B \left|\det(J)\right|\right)_{\xi_i,\eta_i} \tag{18}$$

evaluation of f_{ve}

we note

$$\vec{f} = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} \tag{19}$$

In the case of the gravity field we have :

$$\vec{f} = \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} \tag{20}$$

The discretized volume force writes as follow :

$$f_{ve} = \int_{\Omega_e} \begin{bmatrix} \dots \\ N_k(\xi,\eta) f_{kx} \\ N_k(\xi,\eta) f_{ky} \\ \dots \end{bmatrix} \left|\det(J)\right| d\Omega = \sum_{npi} w_i \begin{bmatrix} \dots \\ N_k(\xi_i,\eta_i) f_{kx} \\ N_k(\xi_i,\eta_i) f_{ky} \\ \dots \end{bmatrix} \left|\det(J_i)\right| \tag{21}$$

evaluation of f_{se}

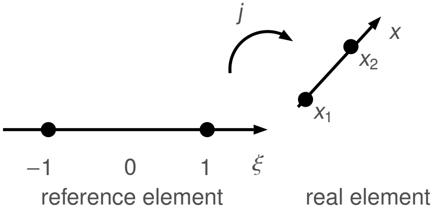
we note

$$\vec{t} = \begin{bmatrix} t_x(s) \\ t_y(s) \end{bmatrix} \tag{22}$$

the imposed stress vector at the boundary of the solid. The discretized surface force writes as follow :

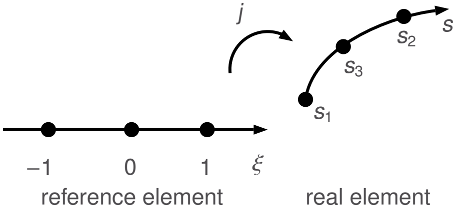
$$f_{se} = \int_{-1}^{1} \begin{bmatrix} \dots \\ N_k(\xi) t_{kx} \\ N_k(\xi) t_{ky} \\ \dots \end{bmatrix} \left|\det(J_s)\right| d\xi = \sum_{npi} w_i \begin{bmatrix} \dots \\ N_k(\xi_i) t_{kx} \\ N_k(\xi_i) t_{ky} \\ \dots \end{bmatrix} \left|\det(J_{si})\right| \tag{23}$$

2 nodes line



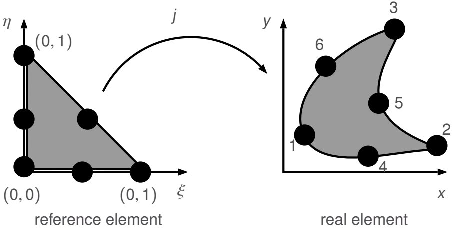
$N_1 = \frac{1}{2}(1-\xi)$	$N_2 = \frac{1}{2}(1+\xi)$

3 nodes line



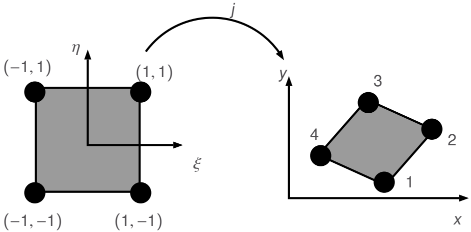
$N_1 = \frac{-1}{2}(1-\xi)\xi$	$N_2 = \frac{1}{2}(1+\xi)\xi$	$N_3 = (1+\xi)(1-\xi)$

6 nodes triangle



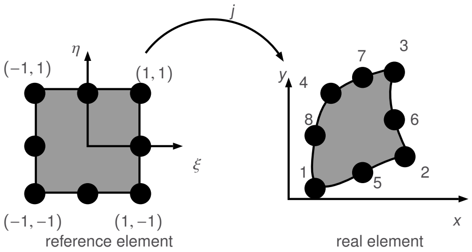
$N_1 = (1-\xi-\eta)(1-2\xi-2\eta)$	$N_4 = 4\xi(1-\xi-\eta)$
$N_2 = \xi(2\xi-1)$	$N_5 = 4\xi\eta$
$N_3 = \eta(2\eta-1)$	$N_6 = 4\eta(1-\xi-\eta)$

4 nodes quadrangle



$N_1 = 1/4\,(1-\xi)\,(1-\eta)$	$N_2 = 1/4\,(1+\xi)\,(1-\eta)$
$N_3 = 1/4\,(1+\xi)\,(1+\eta)$	$N_4 = 1/4\,(1-\xi)\,(1+\eta)$

8 nodes quadrangle



$N_1 = 1/4\,(1-\xi)\,(1-\eta)\,(-\xi-\eta-1)$	$N_2 = 1/4\,(1+\xi)\,(1-\eta)\,(\xi-\eta-1)$
$N_3 = 1/4\,(1+\xi)\,(1+\eta)\,(\xi+\eta-1)$	$N_4 = 1/4\,(1-\xi)\,(1+\eta)\,(-\xi+\eta-1)$
$N_5 = 1/2\left(1-\xi^2\right)(1-\eta)$	$N_6 = 1/2\,(1+\xi)\left(1-\eta^2\right)$
$N_7 = 1/2\left(1-\xi^2\right)(1+\eta)$	$N_8 = 1/2\,(1-\xi)\left(1-\eta^2\right)$

Integration schema formulae I

Gauss schema :

$$\int_{-1}^{1} f(\xi)\, d\xi = \sum_{i=1}^{npi} w_i f(\xi_i) \tag{24}$$

npi	Position of integration points	Weights
1	0	2
2	$\pm 1/\sqrt{3}$	1
3	±0.774596669241483 0	0.555555555555556 0.888888888888889

For $Q4$ and $Q8$ elements we have :

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta)\, d\xi d\eta = \sum_{j=1}^{m}\sum_{i=1}^{n} w_j w_i f(\xi_i, \eta_j) \tag{25}$$

Integration schema formulae II

Hammer's schema for $T3$ or $T6$:

$$\int_0^1 \int_0^{1-\xi} f(\xi, \eta)\, d\xi d\eta = \sum_{i=1}^{n} w_i f(\xi_i, \eta_i) \tag{26}$$

npi	Position of integration points	Weights
1	$(1/3, 1/3)$	1
3	$(1/6, 1/6)$ $(2/3, 1/6)$ $(1/6, 2/3)$	$1/6$ $1/6$ $1/6$

Case of non linear elasticity or elasto-plasticity

in this case, the stress tensor is a non linear (tensorial) function of the strain tensor. If we write the components of the strain and stress tensor in a column vector we get the following general expression :

$$\sigma_i = f_i\left(\varepsilon_1, .., \varepsilon_i, ..., \varepsilon_6\right) \ i = 1...6 \tag{27}$$

As a consequence the global system to solve is non linear :

$$F_{int}\left(U\right) = F_{ext} \tag{28}$$

Iterative Newton's like methods are used to solve such problems.