Elements of Maths for Biology

with solutions to the exercises

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link to the latest version: Moodle

This document is intended for training and future reference. As a reference document, you may find it useful for Biomaths 1 (3BS, Fall semester), for linear algebra (3BIM Biomaths 2, Winter Semester) and for Advanced ODEs (3BIM Biomaths 3, Winter Semester).

When important concepts are encountered for the first time, they are highlighted in **bold** next to their definition. We tried to provides examples that are as complete as possible. This means that they are long, you could probably solve them faster. Exercises are important, they can introduce theory or techniques that will be prove useful. Solutions to the exercises can be found at the end of the documents. Examples and exercises (and their solutions!) will be added regularly, so feel free download the most recent version (here).

Contents

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[1]Based on lecture notes by Laurent Pujo-Menjouet and Samuel Bernard

1 Functions, maps

A **function** is a relation, often denoted f, that associates an element x of a **domain** I, and at most one element y of the **image** J. The domain I and image J are sets. Usually $I, J \in \mathbb{R}$.

A **map** is a relation that associate *each* element of its domain to exactly one element of its image. Maps and functions are related but slightly different concepts. A function f is a map if it is defined for all elements of its domain I. A map is always a function. The term *map* can also be used when the domain or the image are not numbers (Figure 1).

The **graph** of a function f, denoted $\mathcal{G}(f)$ is the set of all pairs $(x, f(x))$ in the $I \times J$ plane. For real-valued functions, the graph is represented as a curve in the Cartesian plane.

Functions are not numbers. Do not confuse

- f the function

- $f(x)$ the evaluation of f at element x; $f(x)$ is an element of the image (usually a number)

- $\mathcal{G}(f)$ the graph of f

Consequently, do not write

- $f(x)$ is increasing... Instead write f is increasing...

- $f(x)$ is decreasing... Instead write f is decreasing...

- $f(x)$ is continuous... Instead write f continuous...

1.1 Some usual maps

- $f : \mathbb{R} \to \mathbb{R}$, with $x \to k$, $k \in \mathbb{R}$ constant; $x \to x$, **identity map**.
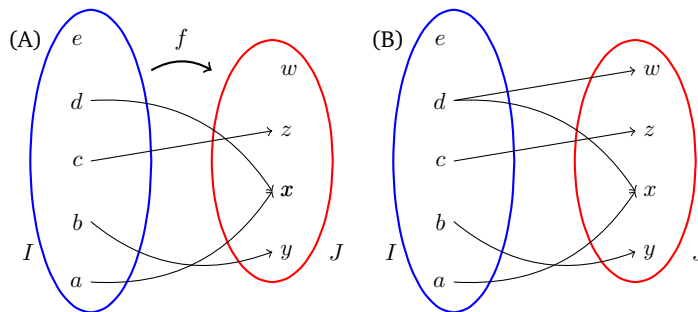
Figure 1. Functions. (A) Function f. (B) Not a function.

constant map

identity map
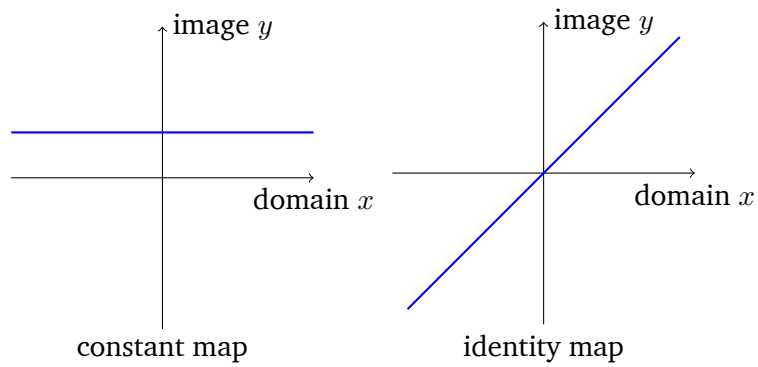
- $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$, with $x \to \frac{1}{x}$, **inverse**.
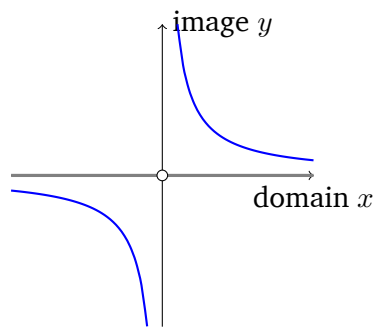
- $f : \mathbb{R} \to \mathbb{R}$, with $x \to x^2$, parabola; $x \to x^3$, **cubic map**.

parabola

cubic map
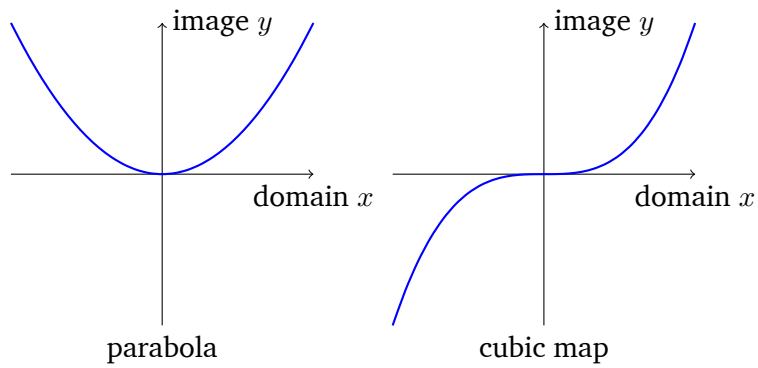
- $f : \mathbb{R}^+ \to \mathbb{R}$, with $x \to \sqrt{x} = x^{\frac{1}{2}}$, **square root**; more generally with $x \to x^{\frac{p}{q}} = \sqrt[q]{x^p}$, **fractional power**.

4

square root

fractional root $(p = 2, q = 3)$
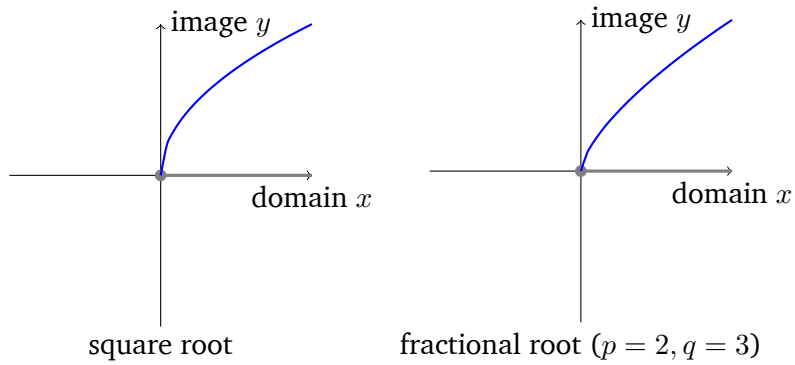
- $f : \mathbb{R}\backslash\{-d/c\} \to \mathbb{R}$, with $x \to \frac{ax+b}{cx+d}$.
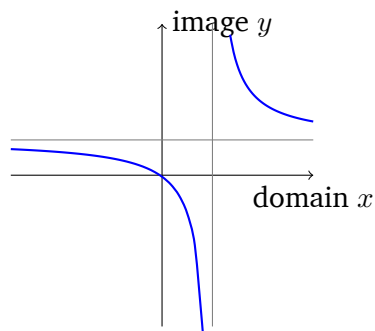
- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \exp(x)$, **exponential**.
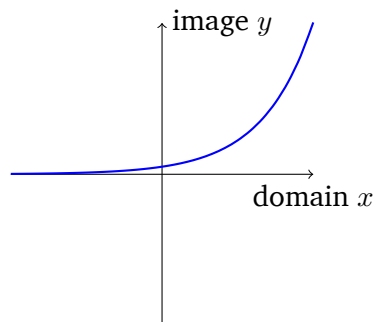
- $f : \mathbb{R}^+\backslash\{0\} \to \mathbb{R}$, with $x \to \ln(x)$, **natural logarithm**.
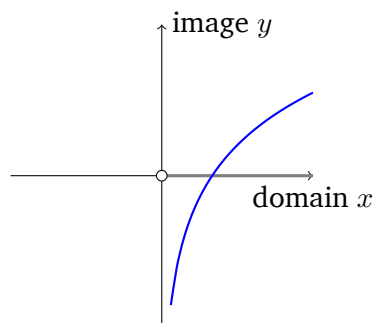
On logarithms: For $a, b > 0$, n positive integer, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^n) = n\ln(a)$, $\ln(a/b) = \ln(a) - \ln(b)$.

- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \cos(x)$, **cosine**; $x \to \sin(x)$, **sine** $x \to \tan(x)$, **tangent**.
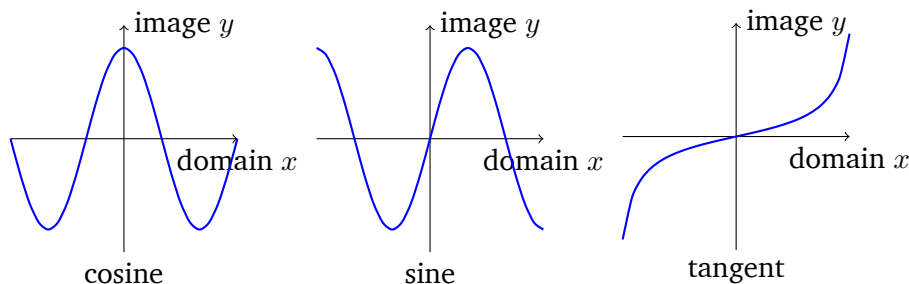
A bit more on trigonometric functions. The diagram below shows the relationship between the sine, cosine and tangent, of an angle $\theta \in [0, 2\pi]$.
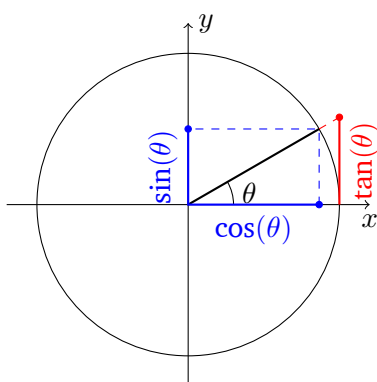
- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$, **hyperbolic cosine**; $x \to \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$, **hyperbolic sine**; $x \to \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, **hyperbolic tangent**.

1.2 Exercises on functions

Exercise 1 (Functions). Draw the graph of the function f defined by $f(x) = \frac{x}{K+x}$, for $x \geq 0$, and $K > 0$. *(Solution to exercise 1)*

Exercise 2 (Functions). Among the following maps, which one are increasing? Decreasing?

- $x \to x(x-3)(x-3)$, $x \in \mathbb{R}$.
- $x \to -x^3$, $x \in \mathbb{R}$.
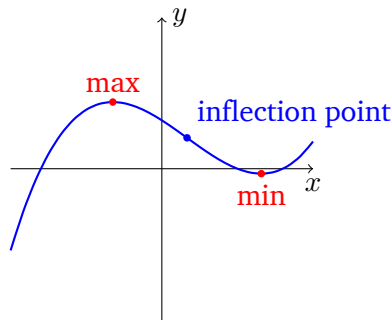
Figure 2. Extrema, inflection points of the polynomial $f(x) = (x + 0.8)(x - 0.5)(x - 0.8)$.

- $x \to \sin(x)$, $x \in \mathbb{R}$.

- $x \to \frac{x^2}{1+x^2}$, $x \geq 0$.

(Solution to exercise 2)

2 Derivatives

We call the **derivative** of the function $f : I \to J$ $(I, J \subset \mathbb{R})$, at point $a \in I$ the limit, if it exists,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The derivative is denoted $f'(a)$. An alternative representation of the limit is obtained by setting $h = x - a$,

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If the derivative exists for all elements $a \in I$, we say that **differentiable** on I.

- If f is differentiable on I, and $f'(x) > 0$, then f is strictly increasing on I.

- If f is differentiable on I, and $f'(x) < 0$, then f is strictly decreasing on I.

However, if f is strictly increasing, it does not mean that $f'(x) > 0$. For example the function f with $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$. Where the derivative exists, we can define the derivative function $f' : I \to \mathbb{R}$ of f.

The **second derivative** of a function f, denoted f'' is the derivative of f', where defined. If $f''(x)$ exists and $f''(x) > 0$ for all $x \in I$, we say that f is **convex** (U-shaped). If $f'(x) = 0$ and $f''(x) > 0$, the point x is a **minimum**. If $f'(x)$ and $f''(x) < 0$, the point x is a **maximum**. Maxima and minima are **extrema**. If $f''(0) = 0$, the point x is an **inflection point** (Figure 2).

2.1 List of common derivatives

The derivative has linear properties. If f and g are differentiable on I, and $a \in \mathbb{R}$,

derivative

differentiable

second
derivative

convex

minimum

maximum

extrema

inflection
point

7

- $(f + g)' = f' + g'.$

- $(af)' = a(f').$

- $(af + g)' = a(f') + g'.$

Let $g : I \to J$ and $f : J \to K$ be two functions. The **composition** of f and g, denoted $f \circ g$, is the function $x \to f(g(x))$, i.e. $f \circ g(x) = f(g(x))$. If f and g are differentiable, the composition $f \circ g$ is also differentiable, and its derivative follows the **rule of composed functions**:

$$\left(f \circ g\right)'(x) = f'(g(x))g'(x).$$

Example 1 Let $f : x \to x^2$ and $g : x \to 3x + 1$ be two differentiable functions, with $f'(x) = 2x$ and $g'(x) = 3$. The derivative of the composed function $f \circ g$ at x is

$$f'(g(x))g'(x) = f'(3x + 1)g'(x) = 2(3x + 1) \cdot 3 = 6(3x + 1) = 18x + 6.$$

The derivative could have been obtained by first computing the composed function $f(g(x)) = (3x + 1)^2 = 9x^2 + 6x + 1$, and then taking the derivative.

Example 2 Compute the derivative of $f : x \to \sin(1/x)$. The function f is composed of a sine and an inverse function. To compute the derivative, we decomposed the function f as $f(x) = g(h(x))$ with $g(x) = \sin(x)$ and $h(x) = 1/x$. The derivatives are $g'(x) = \cos(x)$ and $h'(x) = -1/x^2$. Finally the derivative of f is

$$f'(x) = g'(h(x))h'(x) = \cos(1/x)\left(\frac{-1}{x^2}\right) = -\frac{\cos(1/x)}{x^2}.$$

Example 3 A function $f : I \to I$ is bijective (**invertible**) on I if there exists a function, denoted f^{-1} and called **inverse** of f, such that the compositions are equal $f \circ f^{-1} = f^{-1} \circ f$, and are equal to the identity map. That is, $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$ for all $x \in I$. If f is differentiable and invertible, what is the derivative of f^{-1}?

We apply the derivative to $f(f^{-1})$. Given that $f(f^{-1}(x)) = x$ by definition, we have $\left(f(f^{-1})\right)' = 1$, and

$$\left(f(f^{-1})\right)'(x) = f'(f^{-1}(x))(f^{-1})'(x),$$
$$= 1,$$
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Take for instance $f(x) = x^2$ on $x \in (0, 1]$. The inverse of f is $f^{-1}(x) = \sqrt{x}$. The derivative of f is $f'(x) = 2x$ and the derivative

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(\sqrt{x})}.$$

Example 4 Compute a derivative when the independent variable is in the exponent. To compute the derivative of $f : x \to 2^x$, we need to re-express the function in terms of the natural base e. To do that, we use to properties of the natural logarithm

- For any positive expression y, $y = e^{\ln(y)}$ (ln is the inverse of the exponential function).

- $\ln(a^b) = b\ln(a)$.

Then $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$. The derivative is $\ln(2)e^{x\ln(2)}$. Re-writing in term of base 2, we obtain $f'(x) = \ln(2)2^x$.

Function	Derivative	Note
x^a	ax^{a-1}	$a \in \mathbb{R}$
$\frac{1}{x}$	$\frac{-1}{x^2}$	
$x^{\frac{1}{2}}$	$\frac{1}{2x^{\frac{1}{2}}}$	
$\ln(x)$	$\frac{1}{x}$	
e^x	e^x	
$\cosh(x)$	$\sinh(x)$	
$\sinh(x)$	$\cosh(x)$	
$\cos(x)$	$-\sin(x)$	
$\sin(x)$	$\cos(x)$	
$\tan(x)$	$1 + \tan^2(x) = \frac{1}{\cos^2(x)}$	
$\frac{u(x)}{v(x)}$	$\frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}$	
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	

2.2   Exercises on derivatives

Exercise 3 (Derivatives). Compute the derivatives of the following functions

- $f_1 : x \to \sqrt{\cos x}$.

- $f_2 : x \to \sin(3x + 2)$.

- $f_3 : x \to e^{\cos x}$.

- $f_4 : x \to \ln(\sqrt{x})$.

- $f_5 : x \to 2^{\ln x}$.

(Solution to exercise 3)

3   Taylor series and truncated expansions

If a function f is infinitely differentiable (i.e. the k-th derivative function $f^{(k)}$ is continuous for any integer $k \geq 0$), the Taylor series of f at point a is the series

$$f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + ...$$

A function is **analytic** on an open interval I if and only if its Taylor series converges

analytic
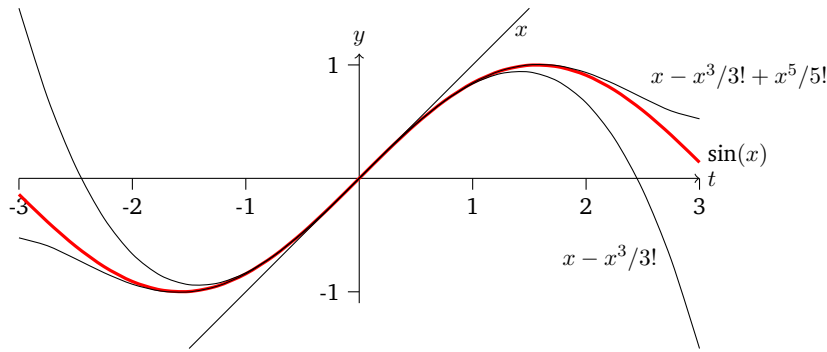
Figure 3. First, third and fifth order expansion of $f(x) = \sin(x)$.

pointwise to the value of the function. Polynomials, exponential and trigonometric function are analytic over all real points. The square root function is not analytic.

The partial sums of the series (or truncated expansion) can be used to approximate a function, and to evaluate it numerically. The k-th order expansion of a function f is the polynomial

$$f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... + \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

Truncated expansions are used in implementations of common mathematical functions in computer programs.

Example 5 The Taylor series of the sine function at point $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n \sin(x)}{dx^n} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

The 3rd-order expansion is the cubic polynomial

$$x - \frac{x^3}{3!}.$$

How good this cubic polynomial approximation to the original sine function? The error is the remainder of the terms of the Taylor series

$$\left| -\frac{x^7}{7!} + \frac{x^9}{9!} - ... \right|.$$

For $|x| < 1$, the error is bounded by $|x|^7/7!$. Given that $7! = 5040$, the error is less than $1/5040 \approx 0.0002$. Truncated expansions are never good approximations when $|x|$ becomes large, because polynomials are not bounded, but the approximation can be quite good over small intervals around the point at which the Taylor series is computed.

It is important to note that even if a function has a Taylor series, the series may not converge to the function.

Example 6 Let the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0. \end{cases}$$

Around 0, all the derivatives exist and are equal to 0. The Taylor series of f at point $x = 0$ is 0, but the function itself is different from the identically zero function.

3.1 Expansion of a function of two variables

For functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}$, the Taylor expansion at point $(x_0, y_0)^t \in \mathbb{R}^2$ is

$$f(x, y) = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$
$$+ \frac{1}{2!}\frac{\partial^2 f(x_0, y_0)}{\partial x^2}(x - x_0)^2$$
$$+ \frac{1}{2!}\frac{\partial^2 f(x_0, y_0)}{\partial y^2}(y - y_0)^2$$
$$+ \frac{2}{2!}\frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}(x - x_0)(y - y_0) + \dots$$

Example 7 The second-order truncated expansion of the function $f(x, y) = ye^{-x}$ at $(0, 0)^t$ is

$$y - xy.$$

3.2 Expansion of a function from $\mathbb{R}^2 \to \mathbb{R}^2$

Functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ have the form

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}.$$

To compute the Taylor series, we need to compute the Taylor series of each function f_1 and f_2. For a first-order expansion at $(x_0, y_0)^t$, we obtain the expansion

$$\begin{pmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f_1(x_0, y_0)}{\partial y}(y - y_0) \\ \frac{\partial f_2(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f_2(x_0, y_0)}{\partial y}(y - y_0) \end{pmatrix}$$

Using vector notation

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix},$$

$$D\boldsymbol{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix},$$

we can write a first-order (or linear) approximation of a function f more compactly with

$$f(x) \approx f(a) + Df(a)(x - a).$$

The approximation is valid only in a neighbourhood of the point a.

4 Integrals and primitives

4.1 Primitives

Let $f : I \to \mathbb{R}$, $(I \subset \mathbb{R})$. A **primitive** F of f on I is a differentiable map such that $F'(x) = f(x)$, $x \in I$.

Example 8 If $f(x) = x^2$, we look for $F(x)$ such that $F'(x) = x^2$. Take $F(x) = x^3/3$, then $F'(x) = 3x^2/3 = x^2$. It also work for $F(x) = x^3/3 + C$, for any constant $C \in \mathbb{R}$.

We denote the primitive as $F(x) = \int f(x)dx$. Primitives are unique up to a constant: if F_1 and F_2 are primitives of f, then $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$, which implies that $F_1'(x) - F_2'(x) = 0$. Therefore the difference between F_1 and F_2, $G : x \to F_1(x) - F_2(x)$ has a zero derivative: $G'(x) = F_1'(x) - F_2'(x) = 0$. This means that G is a constant.

Primitives have linear properties. Let F be a primitive of f, G be a primitive of g and $a, b \in \mathbb{R}$. Then

- $F + G$ is a primitive of $f + g$, or

$$\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx.$$

- aF is a primitive of af, or

$$\int af(x)dx = a \int f(x)dx.$$

Function	Primitive	Note
0	C	$C \in \mathbb{R}$
a	$ax + C$	$a \in \mathbb{R}$
x^a	$\frac{x^{a+1}}{a+1}$	$a \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln|x| + C$	$(a = -1)$, $x \neq 0$
e^x	$e^x + C$	
$\cos(x)$	$\sin(x) + C$	
$\sin(x)$	$-\cos(x) + C$	
$f'(g(x))g'(x)$	$f(g(x)) + C$	

Exercise 4 (Primitives). Compute

$$\int \frac{1}{\sqrt{3x + 5}}dx$$

(Solution to exercise 4)

12

4.2 Integrals

Let $f : [a, b] \to \mathbb{R}$, and F a primitive of f on $[a, b]$. Then the **definite integral** is the value $F(b) - F(a)$, and is denoted

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b.$$

The most important interpretation of the integral is the **area under the curve** (AUC). If $f(x) \geq 0$, $x \in [a, b]$, then

$$\int_a^b f(x)dx$$

is the area under delimited by $f(x)$, x-axis and the axes $x = a$ and $x = b$. Negative values integrate as negative areas.
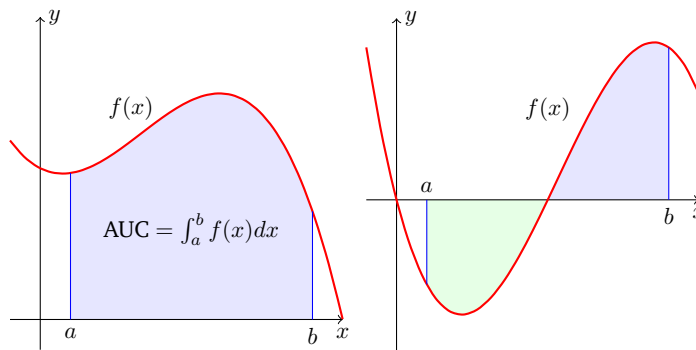
If f is a rate (unit in `something per time`), and x is `time`, then the integral of f has unit `something`. This applies to speed (integral: displacement), production or degradation (integral: concentration), etc.

The integral has the following properties

- Linearity. The integral of a sum is the sum of the integrals.

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

- Negative intervals.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

- Midpoint rule.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Notice that this works even if c is not between a and b.

- The expression

$$\int_a^x f(t)dt = F(x) - F(a) = F_a(x)$$

13

is a function of x. Therefore, for any integrable function f, we have

$$\left(\int_a^x f(x)dx\right)' = f(x).$$

This is the **fundamental theorem of calculus**.

- **Integration by parts**

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx.$$

Integration by part is excessively useful for computing integral beyond simple functions.

Exercise 5 (Integrals). Compute the integral

$$\int_0^1 xe^x dx.$$

(Solution to exercise 5)

Exercise 6 (Integrals). Compute the integral

$$\int_0^\pi \cos x e^x dx.$$

(Solution to exercise 6)

Exercise 7 (Integrals). Compute the integral

$$\int_0^1 \sinh x e^x dx.$$

(Solution to exercise 7)

5 Differential equations in 1D

An **ordinary differential equation** (ODE) of **order** n is a relation (i.e. an equation) between a real variable $t \in I \subset \mathbb{R}$, an unknown function $x : t \to x(t)$ and its derivatives $x', x'', x''', ..., x^{(n)}$ at point t defined by

$$F\big(t, x, x', x'', ..., x^{(n)}\big) = 0,$$

where F *depends* on $x^{(n)}$. In general, $x(t)$ takes values in $\mathbb{R}^N$, i.e. it is a vector. We say that the equation is a **scalar differential equation** if $N = 1$. (The expression $x^{(i)}$ stands for the i-th derivative, not the i-th power.)

The **normal form** of a differential equation of order n is

$$x^{(n)} = f\big(t, x, x', x'', ..., x^{(n-1)}\big).$$

A differential equation is **autonomous** if it does not depend on t, i.e. F has the

ordinary
differential
equation

order

scalar
differential
equation

normal form

autonomous

form

$$F(x, x', x'', ..., x^{(n)}) = 0,$$

A **linear differential equation** is a differential equation for which F is a linear function in $x, x', ...x^{(n)}$. It can be expressed as

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + ...a_1(t)x' + a_0(t)x = g(t),$$

where the coefficients $a_j(t)$ may depend on t, and x and its derivatives $x^{(i)}$ appear only as monomials of degree 1 (that is, linearly). If all coefficients are constants, including g, the linear differential equation is autonomous.

Exercise 8 (Types of differential equations). For the following differential equations, give the order n, and determine whether they are autonomous, linear, and whether they are expressed under their normal form.

1. $x - t + 4tx' = 0$.

2. $(x'')^2 - 2x'tx = 0$.

3. $x^{(3)} + \sin(x') = -5x$.

4. $x^{(4)} - xx' = 0$.

5. $3x'' - 4x' + 6x = 2$.

6. $\ln x' + 3tx = \sqrt{x}$.

(Solution to exercise 8)

A **solution or integral of a differential equation** of order n for t in an interval $I \subset \mathbb{R}$, is a map $x : I-> \mathbb{R}$ that is n times differentiable for all $t \in I$ and satisfies the differential equation.

A **integral curve (or chronic)** is the set of points $(t, x(t))$ for $t \in I$. If $x(t) \in \mathbb{R}^N$, the integral curve is in $\mathbb{R}^{N+1}$. A **trajectory or orbit** is the set of points $x(t)$ for $t \in I$. This is a set in $\mathbb{R}^N$. The space that contains the trajectories is called the **phase space**. The set of all trajectories is called the **phase portrait**.

It is always possible to write an differential equation of order n as a differential equation of order 1, by defining extra variables for the higher-order derivatives.

Example 9 We consider the differential equation $a(t)x'' + b(t)x' + c(t)x = d(t)$. This is a equation of order 2. To reduce it to order 1, let $z_1 = x$ and $z_2 = x'$. Then $x'' = z_2'$ and $z_1' = z_2$. The second-order differential equation can be re-expressed as two first order equations:

$$z_1' = z_2, \quad a(t)z_2' + b(t)z_2 + c(t)z_1 = d(t).$$

Often, the system of first order equations can be re-expressed in normal form, by isolating the variables z_1' and z_2',

$$z_1' = z_2, \text{ and } z_2' = \frac{d(t) - b(t)z_2 - c(t)z}{a(t)}.$$

(We assume that $a(t) \neq 0$.) In general, for the differential equation of order n $F(t, x, x', ..., x^{(n)}) = 0$, with $x : I \to \mathbb{R}^m$, we make a change of variables: $z_1 =$

$x, z_2 = x', ..., z_i = x^{(i-1)}$ until $z_n = x^{(n-1)}$. Each variable $z_i(t) \in \mathbb{R}^m$, so the new vector $z = (z_1, z_2, ..., z_n)^t$ is in $\mathbb{R}^{mn}$. With the change of variables, the differential equation now reads

$$z_1' = z_2,$$
$$z_2' = z_3,$$
$$...,$$
$$z_i' = z_{i+1},$$
$$...,$$
$$F(t, z_1, z_2, ..., z_n, z_n') = 0.$$

Tips on Ordinary differential equations

- The most frequently used differential equations are order 1, and they usually are represented in their normal form: $x' = f(x)$ for autonomous equations, and $x' = f(t, x)$ for non-autonomous equations

- For a scalar, autonomous differential equation $x' = f(x)$ with $x(t) \in \mathbb{R}$, the trajectories are monotonous: if x is a solution, then x is either increasing, decreasing, or constant.

5.1 Finding solutions of differential equations

We consider a **first order scalar differential equation**,

$$a(t)x' + b(t)x = d(t),$$

$t \in I$ and $a(t) \neq 0$ on I, $a(t)$ and $b(t)$ continuous on I. If $d(t) = 0$, we call the equation homogeneous, and

$$a(t)x' + b(t)x = 0.$$

First order scalar ODE. Homogeneous case, first method. Write the equation in normal form,

$$x' = -\frac{b(t)}{a(t)}x.$$

The solution x is either the constant $x = 0$, or $x(t) \neq 0$ for all $t \in I$. We know that because of uniqueness of solutions, which implies that trajectories cannot cross. If $x = 0$ we are done, so we can assume that $x(t) \neq 0$. Dividing the equation by x

$$\frac{x'}{x} = -\frac{b(t)}{a(t)}.$$

The terms on both sides are functions of t. We can compute their primitives

$$\int \frac{x'}{x}dt = -\int \frac{b(t)}{a(t)}dt.$$

The integrand of the left-hand side is x'/x. This is a very common form called **log-derivative** and admits the primitive $\ln|x|$. The right-hand side does not necessarily have a close form, and we leave it as it is. With the integration constant, we obtain implicit solution for x

$$\ln|x| = -\int \frac{b(t)}{a(t)}dt + K.$$

We would like an explicit solution x,

$$|x| = e^{-\int \frac{b(t)}{a(t)}dt + K}.$$

Therefore,

$$x = \pm e^K e^{-\int \frac{b(t)}{a(t)}dt}.$$

Notice that t is a variable of integration, and does not exist outside the integral, and can be replaced by any other variable. To have x as as function of t, we must define the bounds of the integral. When the domain of definition of t is the interval $I = [t_0, t_1], t_0 < t_1$, the solution $x(t)$ is obtained by integrating from t_0 to t,

$$x = \pm e^K e^{-\int_{t_0}^{t} \frac{b(u)}{a(u)}du},$$

where we have replaced the variable of integration by u. Then the definite integral is a function of t. The constant K is determined by the **initial condition** on x at t_0, $x(t_0) = x_0 \in \mathbb{R}$,

$$x(t_0) = \pm e^K e^{-\int_{t_0}^{t_0} \frac{b(u)}{a(u)}du} = \pm e^K = x_0.$$

The constant $K = \operatorname{sign} x_0 \ln x_0$. The fixed bound at t_0 is arbitrary, we could have chosen any other time of reference. However, it is very common to consider differential equations for which we know the value of the solution at the initial time t_0. The problem of solving a differential equation with a given initial condition is called an **initial value problem** (IVP) or a **Cauchy problem**.

First order scalar ODE. Homogeneous case, second method. Assume that $a(t) \neq 0$, and write the equation as

$$x' + \frac{b(t)}{a(t)}x = 0.$$

Multiply the equation by the term $e^{\int \frac{b(t)}{a(t)}dt}$,

$$x'e^{\int \frac{b(t)}{a(t)}dt} + \frac{b(t)}{a(t)}xe^{\int \frac{b(t)}{a(t)}dt} = 0.$$

Notice that the first term has the form $x'f$ and the second term the form xf', where the term f is

$$f = e^{\int \frac{b(t)}{a(t)}dt}.$$

The left-hand-side of the resulting equation, $x'f + xf'$, is the derivative of the product xf, so the differential equation can be integrated,

$$xe^{\int \frac{b(t)}{a(t)}dt} = K,$$

$$x = Ke^{-\int \frac{b(t)}{a(t)}dt}.$$

The constant K is determined by a condition set on the solution, as in the first method.

Exercise 9 (Differential equations). Solve the equation

$$2x' + 6x = 0, \quad x(0) = 1.$$

(Solution to exercise 9)

Exercise 10 (Differential equations). Solve the equation

$$x' + \frac{1}{t}x = 0, \quad x(1) = 1.$$

(Solution to exercise 10)

First order scalar ODE. Heterogeneous case. We now consider the more general differential equation

$$a(t)x' + b(t)x = d(t).$$

Using the strategy from the second method for the homogeneous case above, we divide the equation by $a(t)$, again assuming that $a(t) \neq 0$, and then multiply by $e^{\int \frac{b(t)}{a(t)}dt}$. The resulting equation now has a non-zero right-hand-side,

$$x'e^{\int \frac{b(t)}{a(t)}dt} + \frac{b(t)}{a(t)}xe^{\int \frac{b(t)}{a(t)}dt} = \frac{d(t)}{a(t)}e^{\int \frac{b(t)}{a(t)}dt}.$$

Nevertheless, the left-hand-side is still of the form $x'f + xf'$, and the right-hand-side only depends on t. By integrating both sides, we obtain

$$xe^{\int \frac{b(t)}{a(t)}dt} = \int \frac{d(t)}{a(t)}e^{\int \frac{b(t)}{a(t)}dt}dt + K.$$

The solution for x is then

$$x = \left(\int \frac{d(t)}{a(t)}e^{\int \frac{b(t)}{a(t)}dt}dt + K \right)e^{-\int \frac{b(t)}{a(t)}dt}.$$

Example 10 We consider the differential equation

$$x' + 3x = 1 + \sin(t), \quad x(0) = x_0 > 0.$$

The coefficients $a(t) = 1, b(t) = 3, d(t) = 1 + \sin(t)$. The term $e^{\int \frac{b(t)}{a(t)}dt} = e^{\int 3dt} = e^{3t}$. The general solution is

$$x(t) = \left(\int (1 + \sin(t))e^{3t}dt + K \right)e^{-3t},$$

$$= \frac{1}{3} + \frac{1}{10}\left(\sin(t) - \cos(t) \right) + Ke^{-3t}.$$

At $t = 0$, the equation $x(0) = \frac{1}{3} + \frac{1}{10}(0 - 1) + K = x_0$ solve in K as $K = x_0 - \frac{7}{30}$. The solution $x(t)$ is

$$x(t) = \frac{1}{3} + \frac{1}{10}\left(3\sin(t) - \cos(t) \right) + \left(x_0 - \frac{7}{30} \right)e^{-3t}.$$
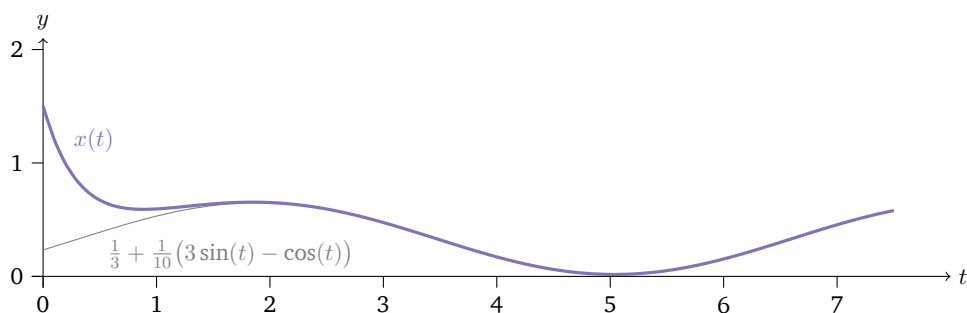
Figure 4. Solution of the initial value problem for the heterogeneous differential equation $x' + 3x = 1 + \sin(x)$.

We now consider a **nonlinear differential equation** of **Bernoulli** type. Bernoulli equations are of the form

$$x' + P(t)x + Q(t)x^r = 0, \quad t \in I \subset \mathbb{R},$$

with continuous functions P, Q, and $r \in \mathbb{R}$. There is no general method to solve nonlinear differential equations, but is can be done is particular cases. If $r = 0$ or $r = 1$, the equation is linear, and we already know how to solve it. Suppose r different from 0 or 1. We will look for positive solutions $x(t) > 0$ on $t \in I$. Dividing by x^r, we get

$$x'x^{-r} + P(t)xx^{-r} + Q(t)x^r x^{-r} = 0,$$
$$x'x^{-r} + P(t)xx^{-r} + Q(t) = 0.$$

We now set an auxiliary variable $u = x^{1-r}$, $(x = u^{1/(1-r)})$. Then

$$u' = (1 - r)x^{1-r-1}x' = (1 - r)x^{-r}x',$$

and, substituting in the differential equation,

$$\frac{1}{1-r}u' + P(t)u + Q(t) = 0,$$
$$u' + (1 - r)P(t)u + (1 - r)Q(t) = 0.$$

We know how to solve this equation; this is a linear equation of the form

$$a(t)u' + b(t)u = d(t),$$

with $a(t) = 1, b(t) = (1 - r)P(t), d(t) = -(1 - r)Q(t)$.

Example 11 Verhulst equation (logistic equation)

$$x' = \mu x\left(1 - \frac{x}{K}\right).$$

We first rewrite the equation in Bernoulli form,

$$x' - \mu x + \mu\frac{x^2}{K} = 0,$$

19
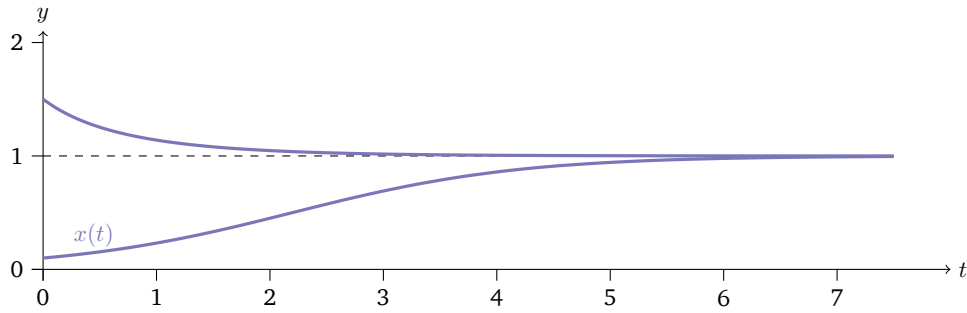
Figure 5. Solution of the initial value problem for the Verhulst equation $x' = \mu x(1 - x/K)$, for $x(0) = 0.1$ and $x(0) = 1.5$. Verhulst equation is a type of Bernoulli equation, and can be solved analytically.

with $P(t) = -1, Q(t) = \mu/K, r = 2$. The auxiliary equation reads

$$u' + (1-2)(-1)\mu u + (1-2)\frac{\mu}{K} = 0. u' + u - \frac{\mu}{K} \qquad = 0.$$

This is a scalar linear equation with $a(t) = 1, b(t) = \mu, d(t) = \mu/K$.. The general solution is

$$u(t) = \left(\int \frac{d(t)}{a(t)} e^{\int b(t)/a(t)dt} dt + C \right) e^{-\int b(t)/a(t)dt},$$

$$\int b(t)/a(t)dt = \int \mu dt = \mu t,$$

$$u(t) = \left(\int \frac{\mu}{K} e^{\mu t} dt + C \right) e^{-\mu t},$$

$$= \left(\frac{\mu}{K} \frac{e^{\mu t}}{\mu} + C \right) e^{-\mu t},$$

$$= \frac{1}{K} + C e^{-\mu t}.$$

The initial condition $u(0) = x_0^{1-r} = x_0^{-1}$,

$$u(0) = \frac{1}{K} + C = \frac{1}{x_0},$$

$$C = \frac{1}{x_0} - \frac{1}{K}.$$

The solution to the original Verhulst equation is $x = u^{-1}$,

$$x(t) = \frac{1}{\frac{1}{K} + \left(\frac{1}{x_0} - \frac{1}{K} \right) e^{-\mu t}}$$

$$= \frac{x_0 K}{x_0 + (K - x_0)e^{-\mu t}}.$$

6   Complex numbers

A complex number is a number that can be expressed in the form $a + ib$, where a and b are real numbers, and the symbol i is called **imaginary unit**. The imaginary

imaginary unit

Figure 6. Rotation in the complex plane. Multiplication by i is a 90 degree counterclockwise rotation.

unit satisfies the equation $i^2 = -1$. Because no real number satisfies this equation, this number is called *imaginary*.

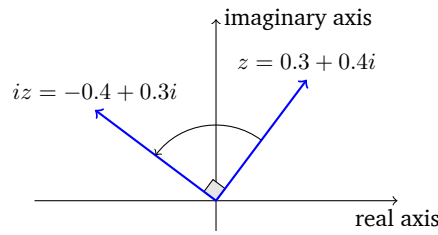For the complex number $z = a + ib$, a is called the **real part** and b is called the **imaginary part**. The real part of z is denoted $\Re(z)$ (\Re in LaTeX) or just Re(z). The imaginary part of z denoted $\Im(z)$ (\Im in LaTeX) or just Im(z). The set of all complex numbers is denoted $\mathbb{C}$ (\mathbb{C} in LaTeX).

We need complex numbers for solving polynomial equations. The fundamental theorem of algebra asserts that a polynomial equation of with real or complex coefficients has complex solutions. These polynomial equations arise when trying to compute the eigenvalues of matrices, something we need to do to solve linear differential equations for instance.

Arithmetic rules that apply on real numbers also apply on complex numbers, by using the rule $i^2 = -1$: addition, subtraction, multiplication and division are associative, commutative and distributive.

Let $u = a + ib$ and $v = c + id$ two complex numbers, with real coefficients a, b, c, d. Then

- $u + v = a + ib + c + id = (a + c) + i(b + d)$.

- $uv = (a + ib)(c + id) = ac + iad + ibc + i^2bd = ac - bd + i(ad + bc)$.

- $\frac{1}{v} = \frac{1}{c+id} = \frac{c-id}{(c-id)(c+id)} = \frac{c-id}{c^2+d^2} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}$.

- $u = v$ if and only if $a = c$ and $b = d$.

  It follows from the rule on i that

- $\frac{1}{i} = -i$. (Proof: $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$.)

Multiplying by the imaginary unit i is equivalent to a counterclockwise rotation by $\pi/2$ (Figure 6)

$$ui = (a + ib)i = ia + i^2b = -b + ia.$$

Let $z = a + ib$ a complex number with real a and b. The **conjugate** of z, denoted $\bar{z}$, is $a - ib$. The conjugate of the conjugate of z is z (*reflection*, Figure 6). The **modulus** of z, denoted $|z|$ is $\sqrt{z\bar{z}}$. The product $z\bar{z} = (a+ib)(a-ib) = a^2+b^2+i(-ab+ab) = a^2+b^2$. The modulus is the complex version of the absolute value, for if z (i.e. $b = 0$), $|z| = \sqrt{a^2} = |a|$. It is always a real, positive number, and $|z| = 0$ if and only if $z = 0$.
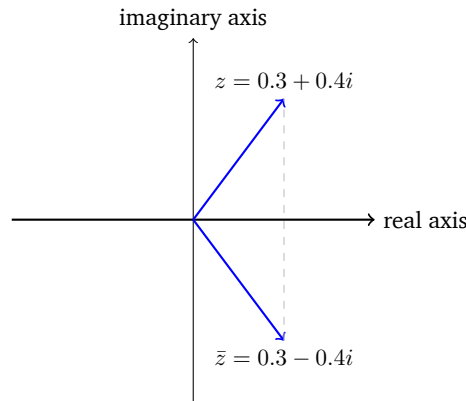
Figure 7. The conjugate of a complex z is the complex $\bar{z}$ obtained by a reflection with respect to the real axis.

The modulus also has the property of being the *length* of the complex number z, if a and b are the sides of a rectangular triangle, then $|z|$ is the hypotenuse.

When simplifying a ratio involving a complex v at the denominator, it is important to convert it to a real number by multiplying the ratio by $\bar{v}/\bar{v}$. For instance, if $v \neq 0$,

$$\frac{u}{v} = \frac{u\bar{v}}{u\bar{v}} = \frac{u\bar{v}}{|v|^2}.$$

The denominator $|v|^2$ is always a positive real number.

By allowing complex values, nonlinear functions of real numbers like exponential, logarithmic and trigonometric functions can have their domain extended to all real and complex numbers. The most useful extension is the exponential function. Recall that the exponential function e^x, where $e \approx 2.71828$ is Euler's constant, satisfies the relation $e^{x+y} = e^x e^y$. This remains true for complex numbers. The **Euler's formula** relates the exponential of a imaginary number with trigonometric functions. For a real number y,

$$e^{iy} = \cos(y) + i\sin(y).$$

Euler's formula

Therefore, for any complex number $z = a + ib$, the exponential

$$e^z = e^{a+ib} = e^a e^{ib} = e^a\big(\cos(b) + i\sin(b)\big).$$

Tips on complex numbers

- If x is real, ix is pure imaginary. If y is imaginary, iy is real.

- $|i| = 1$. For any real θ, $|e^{i\theta}| = 1$.

- $|z_1 z_2| = |z_1||z_2|$.

- In particular, $|iz| = |i||z| = |z|$. (Multiplying by i is a rotation in the complex plane, it does not change the modulus.)
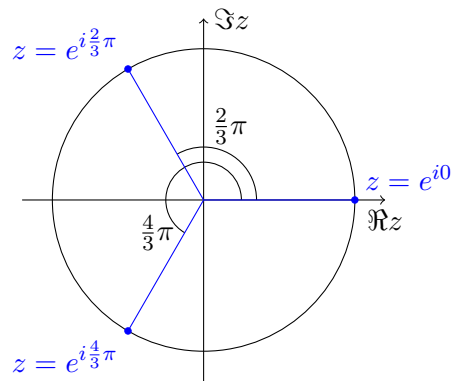
Figure 8. The roots of the polynomial $z^3 - 1$.

6.1 Roots of a complex number

For complex numbers, the equation $z^n = 1$ has n solutions. They are called the **roots of unity**. For $n = 2$, we have the well-known roots $z = \pm 1$, which are real. What are the roots of $z^3 = 1$? To find them, we express z in polar coordinates: $z = re^{i\theta}$. Then

$$z^3 = (re^{i\theta})^3 = r^3 e^{i3\theta} = 1.$$

roots of unity

The equation implies that z has modulus 1, so $r = 1$. The remaining term $e^{i3\theta} = 1$ implies that 3θ is a multiple of 2π because $e^{i\omega} = 1$ if and only if $\omega = 2k\pi$, for some integer k. Therefore $\theta = \frac{2}{3}k\pi$, for $k = 0, 1, 2, ...$. How many distinct points do we have? Clearly, $k = 3$ is equivalent to $k = 0$: $e^{i\frac{2}{3}3\pi} = e^{i2\pi} = ei0$. In the same way $k = 4$ is equivalent to $k = 1$, and so on. Therefore, there are exactly three distinct solutions for θ: $0, \frac{2}{3}\pi, \frac{4}{3}\pi$ (Figure 8).

6.2 Exercises on complex numbers

Exercise 11 (Complex numbers). Let $z = 2 + 3i$. Compute the values of $\bar{z}$, $|z|$, $|\bar{z}|$ (compare with $|z|$), z^2, $\Re(\bar{z})$, $\Im(\bar{z})$, $\frac{z+\bar{z}}{2}$, $\frac{z-\bar{z}}{2}$, $-z$, iz. *(Solution to exercise 11)*

Exercise 12 (Complex numbers). Any complex number can be represented in **polar form**: $z = r(\cos(\theta) + i\sin(\theta))$.

polar form

- Show that $|z| = r$

- Show that $z = re^{i\theta}$

- Conclude that for any complex number z, $|z| = 1$ if and only if z can be expressed as $z = e^{i\theta}$ for a real θ.

(Solution to exercise 12)

Exercise 13 (Euler formula). Using Euler's formula, show that $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a + b)$. *(Use the property that $e^{ia+ib} = e^{ia}e^{ib}$ and apply Euler's*

Formula). *(Solution to exercise 13)*

Exercise 14 (Euler formula). Show Euler's identity: $e^{i\pi} = -1$. *(Solution to exercise 14)*

Exercise 15 (Roots of unity). What are the roots of the equation $z^6 = 1$? *(Solution to exercise 15)*

Exercise 16 (Complex numbers). For a complex z, find necessary and sufficient conditions for e^{zt}, $t > 0$, to converge to 0. *(Solution to exercise 16)*

Exercise 17 (Complex numbers). Let the complex number $z = a + ib$ with real a and b. Compute $\sqrt{z}$ (that is, express $s = \sqrt{z}$ as $s = \alpha + i\beta$, with real α and β) *(Solution to exercise 17)*
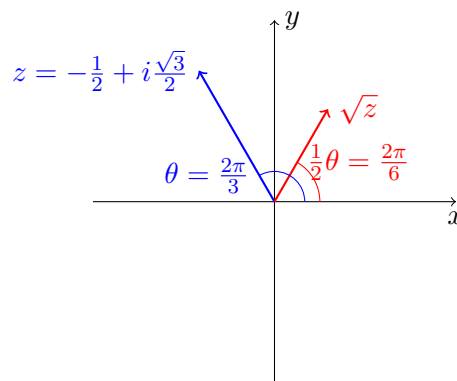
7   Matrices in dimension 2

7.1   Eigenvalues of a 2×2 matrix

A 2×2 matrix A is an array with 2 rows and 2 columns:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Usually, the **coefficients** a, b, c, d are real numbers. The **identity** matrix is the matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The **determinant** of A, denoted $\det A$ or $|A|$ is the number $ad - bc$. The **trace** of A, denoted $\operatorname{tr} A$, is the sum of the main **diagonal** of A: $a + d$.

The **characteristic polynomial** of A is the second order polynomial in λ obtained by computing the determinant of the matrix $A - \lambda I$ (Figure 9),

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = ad - bc - \lambda(a + d) + \lambda^2.$$

The characteristic polynomial $p_A(\lambda)$ of A can be expressed in terms of its determinant and trace:

$$p_A(\lambda) = \det A - \operatorname{tr} A\lambda + \lambda^2.$$

24

Figure 9. Graph of the characteristic polynomial. (*a, green*) polynomial with two negative roots, $p(\lambda) = 0.08 + 0.8\lambda + \lambda^2$. (*b, gray*) polynomial with two complex roots, $p(\lambda) = 0.1 - 0.3\lambda + \lambda^2$. (*c, red*) polynomial with two pos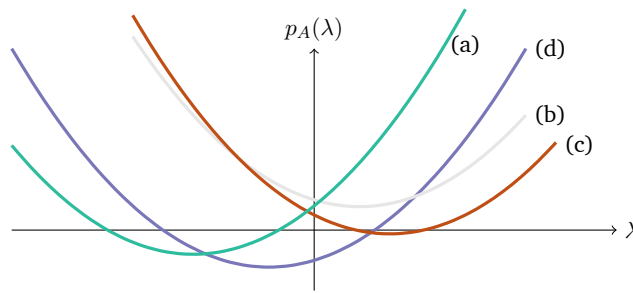itive roots, $p(\lambda) = 0.05 + 0.5\lambda + \lambda^2$. (*d, purple*) polynomial with a negative and a positive root, $p(\lambda) = -0.1 - 0.3\lambda + \lambda^2$.

The **eigenvalues** of A are the roots of the characteristic polynomial. By the fundamental theorem of algebra, we know that the characteristic polynomial has exactly two roots, counting multiple roots. These roots can be real, or complex. The eigenvalues of A are calculated using the quadratic formula:

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right).$$

From this formula, we can classify the eigenvalues of A. Let

$$\Delta = (\operatorname{tr} A)^2 - 4\det A$$

the **discriminant** of the quadratic formula. The two eigenvalues of A are real if and only if $\Delta \geq 0$, i.e. $\operatorname{tr} A)^2 \geq 4\det A$ Then we have the following properties (Figure 10):

1. $\Delta < 0$, complex eigenvalues
    - The two eigenvalues are complex conjugate: $\lambda_1 = \bar{\lambda}_2$
    - Their real part $\Re(\lambda) = \frac{1}{2}\operatorname{tr} A$.

2. $\Delta = 0$, there is a single root of multiplicity 2: $\lambda = \frac{1}{2}\operatorname{tr} A$.

3. $\Delta > 0, \det A > 0$, real, distinct eigenvalues of the same sign.
    - $\operatorname{tr} A > 0$ and $\det A > 0$. Then $\lambda_{1,2}$ are distinct and positive.
    - $\operatorname{tr} A < 0$ and $\det A > 0$. Then $\lambda_{1,2}$ are distinct and negative.

4. $\det A < 0$, real distinct eigenvalues of opposite sign.
    - $\lambda_1 < 0 < \lambda_2$.

5. $\det A = 0$ one of the eigenvalue is zero, the other eigenvalue is $\operatorname{tr} A$.
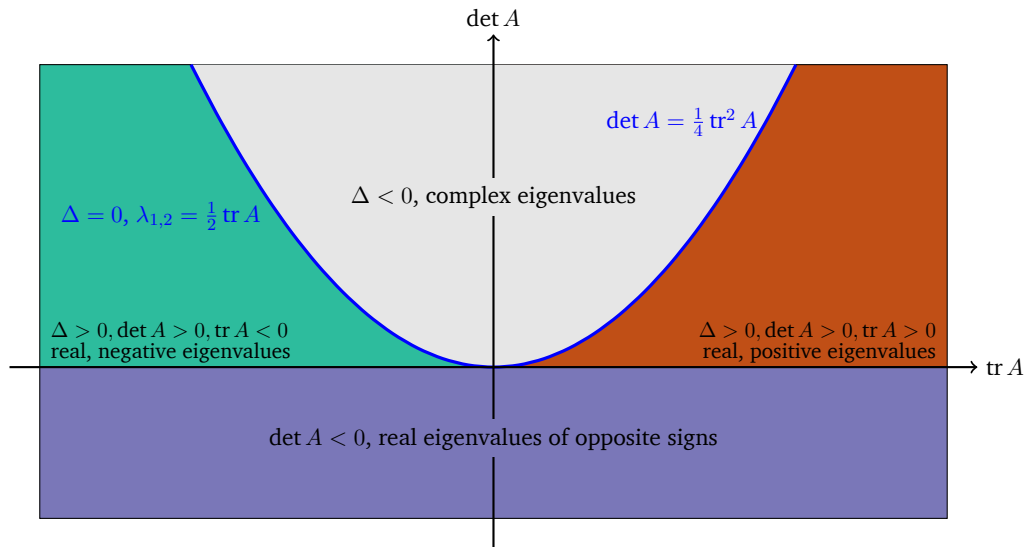
Figure 10. Properties of the eigenvalues of a 2×2 matrix A with respect to $\det A$ and $\operatorname{tr} A$.

7.1.1 Exercises on eigenvalues

Exercise 18 (Eigenvalues). Properties of the eigenvalues of 2×2 matrices. For each 2×2 matrix, compute the determinant, the trace, and the discriminant, and determine whether the eigenvalues are real, complex, distinct, and the sign (negative, positive, or zero) of the real parts.

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 2 \\ 1/2 & 2 \end{pmatrix}.$$

(Solution to exercise 18)

7.2 Matrix-vector operations

A matrix defines a linear transformation between vector spaces. Given a vector x, the product Ax is vector composed of linear combinations of the coefficients of x. For a matrix 2×2, the vector x must be a vector of size 2, and the product Ax is a vector of size two. If $x = (x_1, x_2)^t$ (the ^{t} stands for the transpose, because x must be a column vector), and $A = [a_{ij}]_{i=1,2,\, j=1,2}$, then

$$Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

Successive linear transformations can be accomplished by applying several matrices. Given two matrices A, B, the matrix product $C = AB$ is also a matrix. The matrix C is the linear transformation that first applies B, then A. Matrix product is *not* commutative is general: $AB \neq BA$. (If B means 'put socks on' and A means 'put

shoes on', then BA does not have the expected result!) The product of two matrices $A = [a_{ij}]_{i=1,2,\,j=1,2}$ and $B = [b_{ij}]_{i=1,2,\,j=1,2}$ is

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

The **sum of two matrices** $A + B$ is performed element-wise: $A + B = [a_{ij} + b_{ij}]_{i=1,2,\,j=1,2}$. The **sum of two vectors** is defined similarly. Addition is commutative. Matrix operations are associative and distributive.

sum of two matrices

sum of two vectors

$$A + B = B + A,$$
$$A(B + C) = AB + BC,$$
$$A(BC) = (AB)C.$$

Matrices and vectors can be multiplied by a scalar value (real or complex). **Multiplication by a scalar** is associative, distributive, and commutative. The result of the multiplication by a scalar is to multiply each coefficient of the matrix or vector by the scalar. For example, if λ, μ are scalars,

Multiplication by a scalar

$$\lambda A = A(\lambda I) = A\lambda,$$
$$\lambda(A + B) = \lambda A + \lambda B,$$
$$(\lambda A)B = \lambda(AB) = A(\lambda B),$$
$$(\mu + \lambda)A = \mu A + \lambda A,$$
$$\mu(\lambda A) = (\mu\lambda)A, ...$$

The product between two column vectors is not defined, because the sizes do not match. However, we can define the **scalar product** between two column vectors x, y in the same way matrix product is defined:

scalar product

$$x^t y \equiv x_1 y_1 + x_2 y_2.$$

If the vectors are complex-valued, we need also to conjugate the transposed vector x^t. The conjugate-transpose is called the **adjoint** and is denoted *. Thus, if x is complex-valued, the adjoint x^* is the row vector $(\bar{x}_1, \bar{x}_2)$. The scalar product for complex-valued vectors is denoted $x^* y$. Since this notation also works for real-valued vector, we will use it most of the time.

adjoint

Two vectors are **orthogonal** if their scalar product is 0. In the plane, this means that they are oriented at 90 degree apart. Orthogonal vectors are super important because they can be used to build orthogonal bases that are necessary for solving all sorts of *linear problems*.

orthogonal

7.2.1 Exercises on Matrix-vector and matrix-matrix operations

Exercise 19 (Matrix-vector product). Compute matrix-vector product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

What is the transformation given by this matrix? *(Solution to exercise 19)*

Exercise 20 (Matrix-matrix product). Compute the matrix-matrix product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Can you tell what transformation this is? *(Solution to exercise 20)*

Exercise 21 (Matrix-matrix product). Now compute the product of the same matrices, but in the inverse order

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Compare with the solution found in the previous exercise. What is this transformation? *(Solution to exercise 21)*

Exercise 22 (Matrix construction). Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns the vector $(ax_1, bx_2)^t$. *(Solution to exercise 22)*

Exercise 23 (Matrix construction). Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns the vector $(x_2, x_1)^t$. *(Solution to exercise 23)*

Exercise 24 (Matrix construction). Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns the vector $(x_2, 0)^t$. *(Solution to exercise 24)*

Exercise 25 (Power of a matrix). Compute the successive powers $A, A^2, A^3, ...$, for a diagonal matrix A:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

(Solution to exercise 25)

Exercise 26 (Scalar product). Compute the scalar product x^*y between $x = (1 + 2i, 1 - i)^t$ and $y = (0.5 - i, -0.5)^t$. *(Solution to exercise 26)*

Exercise 27 (Scalar product). Now compute the scalar product y^*x and compare with the result with the previous exercise. *(Solution to exercise 27)*

Exercise 28 (Scalar product). Compute the scalar product between $z = (z_1, z_2)^t$ and itself, if z is a complex-valued vector. What can you say about the result? *(Solution to exercise 28)*

Tips on eigenvalues Some matrices have special shapes that make it easier to compute the determinant, and the eigenvalues. These are called eigenvalue-revealing shapes.

- Diagonal matrices have their eigenvalues on the diagonal.

- **Triangular matrices**, i.e. matrices that have zeros above (lower-triangular matrix) or below (upper-triangular matrix) the main diagonal have also their eigenvalues on the diagonal.

  Triangular matrices

- A matrix with a row or a column of zeros has its determinant equal to zero. This implies that one of its eigenvalues is 0.

---

8 Eigenvalue decomposition

In many applications, it is useful to decompose a matrix into a form that makes it easier to operate complex operations on. For instance, we might want to compute the powers of a matrix A: A^2, A^3, A^4. Multiplying matrices are computationally intensive, especially when the size of the matrix becomes large. The **power of a matrix** is $A^k = AA...A$, k times. The zeroth power is the identity matrix: $A^0 = I$.

The **inverse** of a matrix A, denoted by A^{-1} is the unique matrix such that $AA^{-1} = A^{-1}A = I$. The notation is self-consistent with the positive powers of A. The inverse of a matrix does not always exist. A matrix is **invertible** if and only if its determinant is not 0. If A and B are invertible, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

The **eigenvalue decomposition** is a decomposition of the form $A = XDX^{-1}$, where D is a diagonal matrix, and X is an invertible matrix. If there exists such a decomposition for A, then computing powers of A becomes easy:

$$A^k = (XDX^{-1})^k = XDX^{-1}\,XDX^{-1}...XDX^{-1},$$
$$= XD(X^{-1}X)D(X^{-1}X)D...(X^{-1}X)DX^{-1},$$
$$= XD^kX^{-1}.$$

The eigenvalue decomposition does not always exists, because it is not always possible to find an invertible matrix X. When it exists, though, the columns of the matrix X is composed of the eigenvectors of A. When A is a 2×2 matrix, it is enough to find 2 linearly independent eigenvectors x and y for the matrix

$$X = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

to be invertible.

8.1 Eigenvectors

The **eigenvectors** of a matrix A are the *nonzero* vectors x such that for an eigenvalue λ of A,

$$Ax = \lambda x.$$

If x is an eigenvector, so is any αx for any scalar value α. If there are two linearly independent eigenvectors x and y associated to an eigenvalue, $\alpha x + \beta y$ is also an eigenvector. There is at least one eigenvector for each distinct eigenvalue, but there may be more than one when the eigenvalue is repeated.

Example 12 Distinct, real eigenvalues The matrix

$$A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$$

is upper-triangular; this is one of the eigenvalue-revealing shapes. The eigenvalues are -1 and 1. These are distinct eigenvalues, so each eigenvalue possesses a single eigenvector. The eigenvector x associated to $\lambda_1 = -1$ is found by solving the eigensystem

$$Ax = (-1)x.$$

The unknown quantity x appears on both sides of the equation. We can find a simpler form by noting that multiplying a vector by the identity matrix is neutral: $(-1)x = (-1)Ix$. The eigenproblem becomes

$$Ax = (-1)Ix,$$
$$Ax - (-1)Ix = 0,$$
$$\big(A - (-1)I\big)x = 0,$$

that is, the eigenvector is a nonzero solution of the linear system $(A - \lambda I)x = 0$. In general, if a matrix B is invertible, the only solution to $Bx = 0$ is $x = 0$ (the vector of zeroes). But, by construction, $A - \lambda I$ cannot be invertible if λ is an eigenvalue: its determinant is exactly the characteristic polynomial evaluated at one of its roots, so it is zero. This is why the eigensystem has nonzero solutions. Now, because $A - \lambda I$ is not invertible, this means that a least one of its rows is a linear combination of the others. For 2×2 matrices, this implies that the two rows are colinear, or redundant. For our example, the eigensystem reads

$$\begin{pmatrix} -1 - (-1) & -2 \\ 0 & 1 - (-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

we immediately see that the two rows $(0, -2)$ and $(0, 2)$ are colinear, with a factor -1. This leads to an underdetermined system: $0x_1 + -2x_2 = 0$. The solution is $x_2 = 0$ and we can take x_1 to be any value, save 0. We choose $x = (1, 0)^t$.

For the eigenvalue $\lambda_2 = +1$, the eigensystem reads:

$$\begin{pmatrix} -1 - (+1) & -2 \\ 0 & 1 - (+1) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Again, the second row $(0, 0)$ can be neglected, and the solution is $-2y_1 - 2y_2 = 0$, or $y_1 = -y_2$. It is customary to choose an eigenvector with norm 1. The **norm** of a complex-valued vector $y = (y_1, y_2)^t$ is

norm

$$||y|| = \sqrt{y^*y} = \sqrt{\bar{y}_1 y_1 + \bar{y}_2 y_2} = \sqrt{|y_1|^2 + |y_2|^2}.$$

Here, the eigenvector is $y = (y_1, -y_1)^t$, so $||y|| = \sqrt{|y_1|^2 + |-y_1|^2} = \sqrt{2}\sqrt{|y_1|^2} = \sqrt{2}|y_1|$. Taking $||y|| = 1$ solves $|y_1| = 1/\sqrt{2}$. This means that we could take a negative

or a complex value for y_1, as long as the $|y_1| = 1/\sqrt{2}$. Going for simplicity, we take $y_1 = 1/\sqrt{2}$.

Example 13 Complex eigenvalues

The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is *not* diagonal, so we have to compute the eigenvalues by hand. The trace of A is zero, the determinant is $0 - (1)(-1) = 1$, and the discriminant is -4. A negative discriminant implies complex eigenvalues,

$$\lambda_{1,2} = \frac{1}{2}\left(0 \pm \sqrt{-4}\right) = \pm i.$$

For the eigenvalue $\lambda_1 = +i$, the eigensystem reads:

$$\begin{pmatrix} -(+i) & -1 \\ 1 & -(+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two rows $(-i, 1)$ and $(1, -i)$ should be colinear, but this is not obvious with the complex coefficients. Multiplying the first row by i gives $i(-i, -1) = (-i^2, -i) = (-(-1), -i) = (1, -i)$, the second row, ok. Having confirmed that the system is indeed underdetermined, we can week a solution to $-ix_1 - x_2 = 0$. Solving for $x_2 = -ix_1$, we obtain the eigenvector $x = (x_1, -ix_1)^t$. Normalization of x imposes

$$||x|| = \sqrt{|x_1|^2 + |-ix_1|^2} = \sqrt{|x_1|^2 + |x_1|^2} = \sqrt{2}|x_1| = 1.$$

As in the previous example, we can choose $x_1 = 1/\sqrt{2}$.

The second eigenvectors, associated $\lambda_2 = -i$, solves the eigensystem

$$\begin{pmatrix} -(-i) & -1 \\ 1 & -(-i) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row yields $iy_1 - y_2 = 0$, so $y = (y_1, iy_2)^t$. A normalized eigenvector can be $y = (1/\sqrt{2}, i/\sqrt{2})^t$. We could also have chosen $y = (i/\sqrt{2}, -1/\sqrt{2})^t$.

Example 14 Repeated eigenvalues 1

The matrix

$$\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$$

is lower-trianglar, with repeated eigenvalues on the diagonal, $\lambda_{1,2} = -1$. The eigenvectors associated with -1 satisfy the eigenproblem

$$\begin{pmatrix} -1 - (-1) & 0 \\ 2 & -1 - (-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row vanishes, and the second row means that $x_1 = 0$, leaving for instance $x_2 = 1$, and $x = (0, 1)^t$. There are no other linearly independent eigenvectors. This is not always the case, repeated eigenvalues can have more than one independent eigenvector, as in the next example.

Example 15 Repeated eigenvalues 2

The matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is diagonal, with repeated eigenvalues on the diagonal, $\lambda_{1,2} = -1$. The eigenvectors associated with -1 satisfy the eigenproblem

$$\begin{pmatrix} -1 - (-1) & 0 \\ 0 & -1 - (-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now, the two rows vanished, leaving no condition at all on x_1 and x_2. This means that all the vectors are eigenvectors! How many linearly independent eigenvectors can we find? Vectors of size 2 live in a vector space of dimension 2; we can find at most 2 linearly independent vectors. We can choose for instance the canonical basis: $x = (1, 0)^t$ and $y = (0, 1)^t$.

Tips on eigenvalue decomposition

- A 2×2 matrix (or any square matrix) admits an eigenvalue decomposition if all the eigenvalues are distinct. For 2×2 matrices, eigenvalues are distinct if and only if the discriminant $\Delta \neq 0$.

- If the matrix has a repeated eigenvalue, it will admit an eigenvalue decomposition if the number of (linearly independent) eigenvectors is equal to the number of times the eigenvalue is repeated. The number of eigenvectors is called geometric multiplicity, and the number of repeats is called algebraic multiplicity.

- The eigenproblem should be underdetermined; you should always be able to eliminate at least one row by linear combination. If you cannot, this means that there is a error, possibly an incorrect eigenvalue, or a arithmetic mistake in computing $A - \lambda I$.

- Because eigenvalues are in general complex, the eigenvectors will also be complex.

- The eigenvector matrix X needs to be inverted. When the eigenvectors can be chosen so that they are orthogonal and normalized, the inverse $X^{-1} = X^*$ (i.e. the conjugate transpose of X). Symmetric matrices have orthogonal eigenvalues, so this class of matrices are especially easy to diagonalise.

- Eigenvalue decomposition and invertibility are two different concepts. A matrix can be invertible without admitting an eigenvalue decomposition, and vice versa.

- When a matrix does not admit an eigenvalue decomposition, it still can be triangularised. One such triangularisation is the Jordan decomposition: $A = P(D + S)P^{-1}$, where P is invertible, D is the diagonal matrix of eigenvalues, and S is a **nilpotent** matrix, i.e. a nonzero matrix such that $S^k = 0$ for $k \geq k_0 > 1$.

nilpotent

8.2 Exercises on eigenvalues decomposition

Exercise 29 (Eigenvalue decomposition). Find, if there is any, an eigenvalue decomposition of

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

To compute X^{-1}, you can use the fact that because A is real and symmetrical, the eigenvectors are orthogonal, meaning that $X^{-1} = X^t$, if the eignevectors are normalized. *(Solution to exercise 29)*

9 Linearisation of functions $\mathbb{R}^2 \to \mathbb{R}^2$

Nonlinear systems of ordinary differential equations are used to describe the **dynamics** (evolution in time) of concentration of biochemical species, population densities in ecological systems, or the electrophyiology of neurons.

dynamics

Two-dimensional systems are described by a set of two **ordinary differential equations**, or ODEs,

ordinary differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2),$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2).$$

The variables x_1, x_2 are functions of time: $x_1(t)$, $x_2(t)$, and f_1, f_2 are the derivatives. We define the two-dimensional vectors $\boldsymbol{x} = (x_1, x_2)^t$ (here we will use **bold** for vectors), and $\boldsymbol{f} = (f_1, f_2)^t$. The ODEs can now be represented in vector format,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}).$$

Here we assume that there exists a point in the 2D plane $\bar{\boldsymbol{x}}$ such that the derivative $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$. This point is called a **steady state** because the derivatives are all zeros; the steady state is therefore a solution to the system of ODE.

steady state

We are interested in how $\boldsymbol{f}$ is behaving around the steady state. To do that we linearize the function $\boldsymbol{f}$ at the steady state. **Linearisation** is a first-order expansion. For a function from $\mathbb{R}^2 \to \mathbb{R}^2$, a first-order expansion around a point $\boldsymbol{x}_0$ is

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \boldsymbol{Df}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)$$

When expanding around a steady state, the constant term $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$. In the second term, $\boldsymbol{Df}$ is a 2×2 matrix, called the Jacobian matrix, and often denoted $\boldsymbol{J}$. The **Jacobian matrix** for the function $\boldsymbol{f}$ is defined as

$$\boldsymbol{J} = \boldsymbol{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

When evaluated at a steady state, the Jacobian matrix can provide information on the dynamics of the nonlinear ODE system. More precisely, the eigenvalues of the Jacobian matrix can determine whether the steady state is stable (attracts solutions) or is unstable. **Linearisation around a steady state means computing the Jacobian matrix at the steady state.**

Example 16 Linearisation around a steady state

The Lotka-Volterra equations is a classical ODE system mathematical biology. The equations reads

$$\frac{dx}{dt} = ax - xy,$$
$$\frac{dy}{dt} = xy - by,$$

for a, b positive constants. The solution vector is $\boldsymbol{x} = (x, y)^t$ and the derivatives are $f_1(x, y) = ax - xy$ and $f_2 = xy - by$. We first look for steady states

$$f_1 = ax - xy = 0, \quad f_2 = xy - by.$$

If x and y are not zero, we have $x = b$ and $y = a$. If $x = 0$, the second equation implies $y = 0$. If $y = 0$, the first equation implies $x = 0$. Therefore there are two steady states, $\bar{\boldsymbol{x}} = (b, a)^t$ and $\hat{\boldsymbol{x}} = (0, 0)^t$.

We have the following derivatives

$$\frac{\partial f_1}{\partial x}(x, y) = a - y,$$
$$\frac{\partial f_1}{\partial y}(x, y) = -x,$$
$$\frac{\partial f_2}{\partial x}(x, y) = y,$$
$$\frac{\partial f_2}{\partial y}(x, y) = x - b,$$

The Jacobian matrix is

$$J = \begin{pmatrix} a - y & -x \\ y & x - b \end{pmatrix}.$$

Evaluated at the steady state $\bar{\boldsymbol{x}} = (b, a)^t$ and $\hat{\boldsymbol{x}} = (0, 0)^t$, the Jacobian matrices are

$$J(\bar{\boldsymbol{x}}) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}, \quad J(\hat{\boldsymbol{x}}) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}.$$

9.1 Exercises on linearisation

Exercise 30 (Linearisation). Let the function $\boldsymbol{f} = (f_1, f_2)^t$, with

$$f_1(x, y) = -dx + x\exp(-axy), \quad f_2(x, y) = x - y,$$

$d < 1$, a, d positive. Find the steady states (by solving the equations $f_1 = 0$, $f_2 = 0$). Compute the Jacobian matrix, and evaluate the Jacobian matrix at each steady state. *(Solution to exercise 30)*

Exercise 31 (Jacobian matrices). Compute the Jacobian matrices of each of the following functions of (x, y). All parameters are constants. You do not need to compute the steady states just the matrices.

- van der Pol oscillator

$$f_1(x, y) = \mu\big((1 - x^2)y - x\big), \ f_2(x, y) = y.$$

- Two-compartment pharmacokinetics

$$f_1(x, y) = a - k_{12}x + k_{21}y - k_1 x, \ f_2(x, y) = k_{12}x - k_{21}y.$$

- SI epidemiological model

$$f_1(x, y) = -\beta xy, \ f_2(x, y) = \beta xy - \gamma y.$$

(Solution to exercise 31)

10 Solution of systems of linear differential equations in dimension 2

Linear differential equations have linear derivative parts, which can be represented in matrix-vector format

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t),$$

for a vector $\boldsymbol{x}$ square matrix $\boldsymbol{A}$. For initial conditions $\boldsymbol{x}(t) = \boldsymbol{x}_0$, the **solution of the linear system of ODEs** is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

If we have at our disposal an eigenvalue decomposition of $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$, the **exponential of the matrix** is

$$e^{\boldsymbol{A}t} = \boldsymbol{X}e^{\boldsymbol{D}t}\boldsymbol{X}^{-1},$$
$$= \boldsymbol{X}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\boldsymbol{X}^{-1}.$$

exponential of
the matrix

Therefore, the long-time behavior of the exponential is controlled by the eigenvalues $\lambda_{1,2}$.

Example 17 Solution of a linear system of ODEs

Consider the linear system of ODEs given by the Lotka-Volterra model linearised at its nonzero steady state $\bar{\boldsymbol{x}} = (b, a)^t$ is

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{1}$$

This system approximates the nonlinear version near the steady state. In this linear system, variables (x, y) are deviations from the steady state; their solutions are "centered" around 0. To solve this linear system, we will diagonalise the matrix

$$A = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}.$$

The goal is to go slowly through every step once for this system. In general it is not necessary to solve the system completely by hand; knowledge of the eigenvalues is often sufficient in many applications.

We have $\det A = 0 - a(-b) = ab > 0$, $\operatorname{tr} A = 0$ and $\Delta = 0 - 4ab = -4ab < 0$. The eigenvalues are therefore complex conjugates: $\lambda_{1,2} = \pm i\sqrt{ab}$. Distinct eigenvalues means that A is diagonalisable. The eigenvector associated to $\lambda_1 = i\sqrt{ab}$ is given by the system

$$\left(\begin{array}{cc|c} -i\sqrt{ab} & -b & 0 \\ a & -i\sqrt{ab} & 0 \end{array} \right)$$

We have from the first row $-i\sqrt{ab}x = by$. Letting $x = b$ and $y = -i\sqrt{ab}$, we obtain the non-normalized eigenvector $\tilde{x}_1 = (b, -i\sqrt{ab})^t$. Normalization is done by dividing by

$$||\tilde{x}|| = \sqrt{b^2 + (-i\sqrt{ab})^2} = \sqrt{b^2 + ab},$$

to obtain the first eigenvector

$$x = \begin{pmatrix} \frac{b}{\sqrt{b^2+ab}} \\ \frac{-i\sqrt{ab}}{\sqrt{b^2+ab}} \end{pmatrix} = \begin{pmatrix} \frac{b}{\sqrt{b}\sqrt{b+a}} \\ \frac{-i\sqrt{a}\sqrt{b}}{\sqrt{b}\sqrt{b+a}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{b}}{\sqrt{b+a}} \\ \frac{-i\sqrt{a}}{\sqrt{b+a}} \end{pmatrix}.$$

The second eigenvector is computed the same way (watch out for the slightly different signs!). The eigenproblem for the eigenvalue $\lambda = -i\sqrt{ab}$ is

$$\left(\begin{array}{cc|c} +i\sqrt{ab} & -b & 0 \\ a & +i\sqrt{ab} & 0 \end{array} \right)$$

Given that the only change is $-i \rightarrow +i$, the second eigenvector is

$$x_2 = \left(\begin{array}{c} \frac{\sqrt{b}}{\sqrt{b+a}} \\ \frac{i\sqrt{a}}{\sqrt{b+a}} \end{array} \right).$$

The solution to the linear ODE is

$$\left(\begin{array}{c} x(t) \\ y(t) \end{array} \right) = \boldsymbol{X} e^{\boldsymbol{D} t} \boldsymbol{X}^{-1} \left(\begin{array}{c} x_0 \\ y_0 \end{array} \right),$$

with

$$\boldsymbol{X} = \frac{1}{\sqrt{b+a}} \left(\begin{array}{cc} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{array} \right), \quad \boldsymbol{D} = \left(\begin{array}{cc} +i\sqrt{ab} & 0 \\ 0 & -i\sqrt{ab} \end{array} \right)$$

The **inverse of a** 2×2 **matrix** with coefficients a, b, c, d is

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \frac{1}{ad - bc} \left(\begin{array}{cc} d & -b \\ -c & a \end{array} \right).$$

This is conditional to $\det = ad - bc \neq 0$, of course. With this formula, the inverse of $\boldsymbol{X}$ is

$$\boldsymbol{X}^{-1} = \frac{1}{\sqrt{b+a}} \frac{1}{\det \boldsymbol{X}} \left(\begin{array}{cc} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{array} \right).$$

The determinant $\det \boldsymbol{X} = \frac{i\sqrt{b}\sqrt{a}}{b+a} + \frac{i\sqrt{a}\sqrt{b}}{b+a} = 2i\frac{\sqrt{ab}}{b+a}$. The inverse reduces to

$$\frac{1}{\sqrt{b+a}} \frac{a+b}{2i\sqrt{ab}} \left(\begin{array}{cc} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{array} \right) = \frac{-i\sqrt{b+a}}{2\sqrt{ab}} \left(\begin{array}{cc} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{array} \right) = \frac{\sqrt{b+a}}{2\sqrt{ab}} \left(\begin{array}{cc} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{array} \right).$$

We have now obtained the eigenvalue decompostion of $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$. To solve the linear ODE, we need to compute the product

$$\left(\begin{array}{c} x(t) \\ y(t) \end{array} \right) = \boldsymbol{X} e^{\boldsymbol{D} t} \boldsymbol{X}^{-1} \left(\begin{array}{c} x_0 \\ y_0 \end{array} \right),$$

$$= \frac{1}{\sqrt{b+a}} \left(\begin{array}{cc} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{array} \right) \left(\begin{array}{cc} e^{i\sqrt{ab}t} & 0 \\ 0 & e^{-i\sqrt{ab}t} \end{array} \right) \frac{\sqrt{b+a}}{2\sqrt{ab}} \left(\begin{array}{cc} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{array} \right) \left(\begin{array}{c} x_0 \\ y_0 \end{array} \right),$$

$$= \frac{1}{2\sqrt{ab}} \left(\begin{array}{cc} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{array} \right) \left(\begin{array}{cc} e^{i\sqrt{ab}t} & 0 \\ 0 & e^{-i\sqrt{ab}t} \end{array} \right) \left(\begin{array}{cc} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{array} \right) \left(\begin{array}{c} x_0 \\ y_0 \end{array} \right),$$

$$= \frac{1}{2\sqrt{ab}} \left(\begin{array}{cc} \sqrt{b}e^{i\sqrt{ab}t} & \sqrt{b}e^{-i\sqrt{ab}t} \\ -i\sqrt{a}e^{i\sqrt{ab}t} & i\sqrt{a}e^{-i\sqrt{ab}t} \end{array} \right) \left(\begin{array}{c} \sqrt{a}x_0 + i\sqrt{b}y_0 \\ \sqrt{a}x_0 - i\sqrt{b}y_0 \end{array} \right).$$
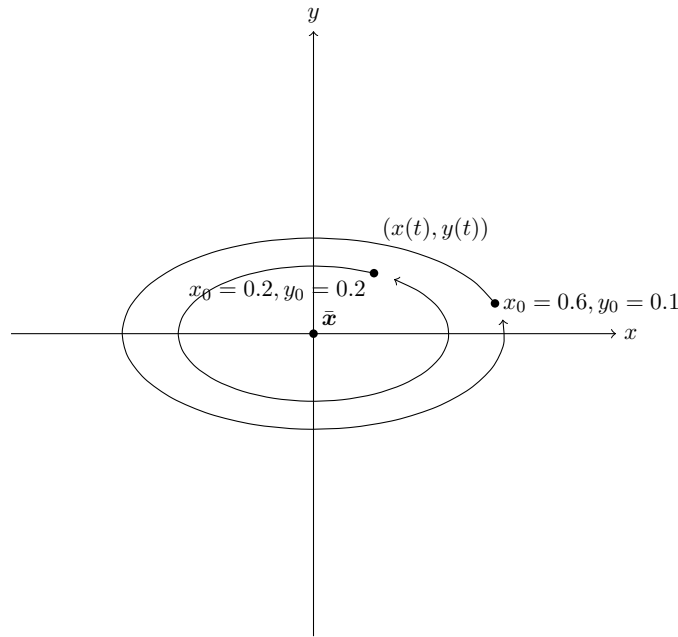
Figure 11. Solution of the linear system of ODEs (1), with $a = 0.1$, $b = 0.4$.

To simplify the last steps of the calculation, we will introduce the following notation. Using Euler's formula, we have $e^{\pm i\sqrt{ab}t} = \cos(\sqrt{ab}t) \pm i\sin(\sqrt{ab}t)$. Let the parameters c, s, C_1 and C_2 be $c = \cos(\sqrt{ab}t)$, $s = \sin(\sqrt{ab}t)$, and $C_1 = \sqrt{a}x_0 + i\sqrt{b}y_0$, $C_2 = \sqrt{a}x_0 - i\sqrt{b}y_0$. The solution now reads in a more compact manner:

$$
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}e^{i\sqrt{ab}t}C_1 + \sqrt{b}e^{-i\sqrt{ab}t}C_2 \\ -i\sqrt{a}e^{i\sqrt{ab}t}C_1 + i\sqrt{a}e^{-i\sqrt{ab}t}C_2 \end{pmatrix},
$$

$$
= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}(c+is)C_1 + \sqrt{b}(c-is)C_2 \\ -i\sqrt{a}(c+is)C_1 + i\sqrt{a}(c-is)C_2 \end{pmatrix},
$$

$$
= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}c(C_1 + C_2) + i\sqrt{b}s(C_1 - C_2) \\ \sqrt{a}s(C_1 + C_2) + i\sqrt{a}c(-C_1 + C_2) \end{pmatrix},
$$

$$
= \frac{1}{2\sqrt{ab}} \begin{pmatrix} 2\sqrt{ab}\cos(\sqrt{ab}t)x_0 - 2b\sin(\sqrt{ab}t)y_0 \\ 2a\sin(\sqrt{ab}t)x_0 + 2\sqrt{ab}\cos(\sqrt{ab}t)y_0 \end{pmatrix},
$$

$$
= \begin{pmatrix} \cos(\sqrt{ab}t)x_0 - \sqrt{b/a}\sin(\sqrt{ab}t)y_0 \\ \sqrt{a/b}\sin(\sqrt{ab}t)x_0 + \cos(\sqrt{ab}t)y_0 \end{pmatrix}.
$$

And that's it! We have obtained a solution to the linear ODE (Figure 11).

11 Solutions to the exercises

Solution to exercise 1 The graph of f, $f(x) = \frac{x}{K+x}$, is
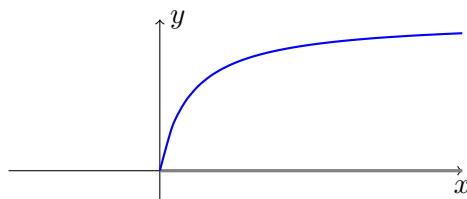
Solution to exercise 2 The map $x \to -x^3$ is decreasing over $x \in \mathbb{R}$. The map $x \to \frac{x^2}{1+x^2}$ is increasing for $x \geq 0$. The other maps are not strictly increasing or decreasing.

Solution to exercise 3 $f_1'(x) = -\frac{\sin x}{2\sqrt{\cos x}}$. $f_2'(x) = 3\cos(3x+2)$. $f_3'(x) = -\sin(x)e^{\cos x}$. $f_4'(x) = \frac{1}{2x}$. $f_5'(x) = \frac{\ln 2}{x} 2^{\ln x}$.

Solution to exercise 4 By the change of variable $u = 3x+5, du = 3dx$, we have

$$\int \frac{1}{\sqrt{3x+5}} dx = \int \frac{1}{\sqrt{u}} \frac{du}{3},$$
$$= \frac{1}{3} \frac{u^{\frac{1}{2}}}{\frac{1}{2}},$$
$$= \frac{2}{3} u^{\frac{1}{2}},$$
$$= \frac{2}{3}(3x+5)^{\frac{1}{2}}.$$

Solution to exercise 5 The integrand is the product of two functions, so let us try integration by parts. Let $g(x) = x$ and $f'(x) = e^x$. Then $g'(x) = 1$ and $f(x) = e^x$. The integral becomes

$$\int_0^1 xe^x dx = e^x x \Big|_0^1 - \int_0^1 e^x dx,$$
$$= e^1 - 0 - e^x \Big|_0^1,$$
$$= e - (e^1 - e^0),$$
$$= 0 + 1,$$
$$= 1.$$

Solution to exercise 6 The integrand is the product of two functions, we try integration by parts. Let $g(x) = \cos x$ and $f'(x) = e^x$. Then $g'(x) = -\sin x$ and $f(x) = e^x$.

The integral becomes

$$\int_0^\pi \cos x\, e^x dx = e^x (\cos x)\Big|_0^\pi - \int_0^\pi e^x(-\sin x)dx,$$
$$= -e^\pi - e^0 - \int_0^\pi e^x(-\sin x)dx,$$
$$= -e^\pi - 1 - \int_0^\pi e^x(-\sin x)dx,$$

We still have a integral of a product to compute. We apply the integral by part once more,

$$= -e^\pi - 1 - \left(e^x(-\sin x)\Big|_0^\pi + \int_0^\pi e^x(\cos x)dx\right).$$
$$= -e^\pi - 1 - \int_0^\pi e^x(\cos x)dx.$$

With this new integration, we just came back to our initial integral. We are looping, and further integration by part will not help us. To break the loop, we use the fact that the initial integral term is present on both side of the equation and solve for it. If

$$I = \int_0^\pi \cos x\, e^x dx,$$

then

$$I = -e^\pi - 1 - I,$$
$$2I = -e^\pi - 1,$$
$$I = \frac{-e^\pi - 1}{2}.$$

Et voilà!

Solution to exercise 7 TODO

Solution to exercise 8

1. linear, non-autonomous, not in normal form

2. non-linear, non-autonomous, not in normal form

3. non-linear, autonomous, not in normal form

4. non-linear, autonomous, not in normal form

5. non-linear, non-autonomous, not in normal form

Solution to exercise 9 The initial condition $x(0) = 1$ tells us to start the integration at $t = 0$. Using the second method with $a(t) = 2$ and $b(t) = 6$, we have

$$x(t) = Ke^{-\int_0^t -\frac{6}{2} du} = Ke^{-\frac{6}{2}t}.$$

The constant K is determined by the initial condition,

$$x(0) = Ke^0 = K = 1.$$

The complete solution is

$$x(t) = e^{-\frac{6}{2}t}.$$

Solution to exercise 10 The initial condition $x(1) = 1$ tells us to start the integration at $t = 1$. Using the second method with $a(t) = 1$ and $b(t) = 1/t$, we have

$$x(t) = Ke^{-\int_1^t -\frac{1}{t} du} = Ke^{-\ln u|_1^t} = Ke^{-\ln t}.$$

The term $e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$. The solution simplifies to

$$x(t) = \frac{K}{t}.$$

The constant K is found with the initial condition $x(1) = K = 1$, for a complete solution $x(t) = \frac{1}{t}$.

Solution to exercise 11 $\bar{z} = 2 - 3i$, $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$, $|\bar{z}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$, we see that $|z| = |\bar{z}|$, $\Re(\bar{z}) = 2$, $\Im(\bar{z}) = -3$, $\frac{z+\bar{z}}{2} = (2 + 3i + (2 - 3i))/2 = 2$, $\frac{z-\bar{z}}{2} = ((2 + 3i - (2 - 3i))/2 = 3i$, $-z = -2 - 3i$, $iz = 2i + 3i^2 = -3 + 2i$.

Solution to exercise 12 The modulus of z is $|z| = \sqrt{r^2 \cos^2(\theta) + r^2 \sin^2(\theta)} = \sqrt{r^2} = r$. From Euler's formula, we have $\cos(\theta) + i\sin(\theta) = e^{i\theta}$, so $z = re^{i\theta}$. Therefore, for any complex number $z = re^{i\theta}$, $|z| = 1$ if and only if $r = 1$.

Solution to exercise 13 All trigonometric identities can be obtained by applying Euler's formula. Here we start from $e^{ia+ib} = \cos(a + b) + i\sin(a + b)$. We only want the real part,

$$\begin{aligned}
\cos(a + b) &= \frac{e^{ia+ib} + e^{-ia-ib}}{2} \\
&= \frac{e^{ia}e^{ib} + e^{-ia}e^{-ib}}{2} \\
&= \frac{(\cos(a) + i\sin(a))(\cos(b) + i\sin(b))}{2} \\
&\quad + \frac{(\cos(a) - i\sin(a))(\cos(b) - i\sin(b))}{2} \\
&= \frac{\cos(a)\cos(b) + i^2\sin(a)\sin(b) + i\cos(a)\sin(b) + i\cos(b)\sin(a)}{2} \\
&\quad + \frac{\cos(a)\cos(b) + i^2\sin(a)\sin(b) - i\cos(a)\sin(b) - i\cos(b)\sin(a)}{2}.
\end{aligned}$$

The mixed cosine-sine terms cancel each other while the other ones add up, resulting in

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Solution to exercise 14 This is a direct application of Euler's formula: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

Solution to exercise 15 The roots must satisfy $e^{i6\theta} = 1$. This means that $\theta = \frac{2}{6}k\pi$, for $k = 0, 1, ..., 5$. There are six distinct roots.

Solution to exercise 16 The exponential converges to zero if and only if $\Re(z) < 0$. A complex number is close to zero if and only if its modulus is close to zero. Therefore, to show that a quantity converges to zero, it is necessary and sufficient to show that its modulus converges to zero. If $z = a + ib$, the exponential $e^{zt} = e^{(a+ib)t} = e^{at}e^{ibt}$. The modulus $|e^{ibt}| = 1$, so $|e^{zt}| = e^{at}$ (no need for absolute values, the exponential of a real number is always positive). The condition for convergence to zero is therefore a condition on the real part of z: $e^{at} \to 0$ when $t \to \infty$ if and only if $a < 0$.

Solution to exercise 17 The square root of a complex number always exists. Express z in polar form $z = re^{i\theta}$, $r \geq 0, \theta \in [0, 2\pi]$. The square root $\sqrt{z} = \sqrt{r}\sqrt{e^{i\theta}} = \sqrt{r}e^{\frac{1}{2}i\theta}$. Using Euler's formula, $\sqrt{z} = \sqrt{r}\cos(\theta/2) + i\sqrt{r}\sqrt{\theta/2}$. That is, the square root is obtained by taking the square root of the modulus r, and dividing the angle (the **argument**) by 2. There is a problem with this solution, because z can also be represented by $re^{i\theta+2\pi}$, giving $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta+\pi}$, which is equivalent to dividing the angle by two in the other direction. We define the **principal square root** as the solution that makes the smallest change in angle: $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta}$ if $\theta \in [0, \pi]$, and $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta+\pi}$ if $\theta \in (\pi, 2\pi]$ To express the solution in terms of the original form of $z = a + ib$, we express the square root $s = \alpha + i\beta$. Then $s^2 = \alpha^2 - \beta^2 + 2i\alpha\beta = z = a + ib$. By identifying the real and imaginary parts, we get two equations: $\alpha^2 - \beta^2 = a$ and $2i\alpha\beta = b$. Denoting the modulus of z by $r = \sqrt{a^2+b^2}$, we can obtain the solutions

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{a+r}, \quad \beta = \text{sign}(b)\frac{1}{\sqrt{2}}\sqrt{-a+r}.$$

argument

principal square root

Solution to exercise 18 TODO

Solution to exercise 19

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

The transformation is a 90 degree counterclockwise rotation.

Solution to exercise 20

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix exchanges the coordinates of a vector, this is a reflection through the axis $x = y$.

Solution to exercise 21

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The pruduct is not the same, the matrices do not commute. The transformation is now a reflection through $y = -x$.

Solution to exercise 22 The matrix is

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Solution to exercise 23 The matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution to exercise 24 The matrix is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solution to exercise 25 The power of a diagonal matrix is a diagonal matrix

$$A^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}.$$

Solution to exercise 26 The scalar product is

$$\begin{aligned} x^* y &= (1 - 2i, 1 + i)(0.5 - i, -0.5)^t \\ &= (1 - 2i)(0.5 - i) + (1 + i)(-0.5) \\ &= 0.5 + 2i^2 - 2(0.5)i - i - 0.5 - 0.5i \\ &= (0.5 - 0.5 + 2i^2) + (-2(0.5) - 1 - 0.5)i \\ &= -2 - 2.5i. \end{aligned}$$

Solution to exercise 27 The scalar product is

$$\begin{aligned} y^* x &= (0.5 + i, -0.5)(1 + 2i, 1 - i)^t \\ &= (0.5 + i)(1 + 2i) + (-0.5)(1 - i) \\ &= 0.5 + 2i^2 + i + 2(0.5)i - 0.5 + 0.5i \\ &= -2 + 2.5i \end{aligned}$$

This is the conjugate: $x^*y = (y^*x)^*$.

Solution to exercise 28 The scalar product $z^*z = (\bar{z}_1, \bar{z}_2)(z_1, z_2)^t = \bar{z}_1 z_1 + \bar{z}_2 z_2 = |z_1|^2 + |z_2|^2$. The scalar product is the square of the norm of the vector z.

Solution to exercise 29 We have $\det A = (-1)(-1) - (2)(2) = 1 - 4 = -3 < 0$, $\operatorname{tr} A = -1 - 1 = -2$, and $\Delta = (-2)^2 - 4(-3) = 4 + 12 = 16$. The eigenvalues are
$$\lambda_{1,2} = \frac{1}{2}(-2 \pm \sqrt{16}) = -1 \pm 2 = 1, -3.$$
The two eigenvalues are distinct, so the matrix A is diagonalisable. The eigenvector associated with the eigenvalue $\lambda_1 = 1$ is solution to the eigenproblem $(A - \lambda_1 I)x = 0$. We look for a solution
$$\begin{pmatrix} -1 - (1) & 2 \\ 2 & -1 - (1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$
$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

(We check that the two rows are colinear.) The first row gives $-2x_1 + 2x_2 = 0$, or $x_1 = x_2$. The norm of the eigenvector $x = (x_1, x_2)^t = \sqrt{x_1^2 + x_2^2} = \sqrt{2}|x_1|$. We choose $x_1 = 1/\sqrt{2}$ to have a normalized eigenvector. We know that the second eigenvector is orthogonal to x, so we can take $y = (1/\sqrt{2}, -1/\sqrt{2})^t$ for the eigenvector associated to $\lambda_2 = -3$. To check that this is indeed an eigenvector, we solve to eigenproblem
$$\begin{pmatrix} -1 - (-3) & 2 \\ 2 & -1 - (-3) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0.$$

The solutions are vectors that satisfy $y_1 = -y_2$; this is the case for y. The matrix X is composed of the column vectors x and y: $X = (x|y)$ and its inverse is
$$X^{-1} = X^t = \begin{pmatrix} x_1^t \\ x_2^t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution to exercise 30 The steady states are found by solving
$$f_1(x, y) = -dx + x \exp(-axy) = 0,$$
$$f_2(x, y) = x - y = 0.$$

From the second equation, we have $x = y$. The first equation is equivalent to $dx = x \exp(-axy)$. We need to distinguish two cases: (i) $x = 0$, and (ii) $x \neq 0$. Case (i) leads to the solution $\boldsymbol{x}^* = (0, 0)^t$, our first steady state. Case (ii) means that we can simplify x in the first equation: $d = \exp(-axy)$. Replacing $y = x$, and solving for

x:

$$d = \exp(-axy),$$
$$d = \exp(-ax^2),$$
$$\ln d = -ax^2,$$
$$-\frac{\ln d}{a} = x^2, \quad (a > 0)$$

The hypothesis $d < 1$ ensures that $\ln d < 0$ and $-\ln d > 0$. There are therefore two real solutions for x:

$$x = \pm\sqrt{-\frac{\ln d}{a}}.$$

The two additional steady states are

$$\bar{x}_{1,2} = \begin{pmatrix} \pm\sqrt{-\frac{\ln d}{a}} \\ \pm\sqrt{-\frac{\ln d}{a}} \end{pmatrix}.$$

The Jacobian matrix of $\boldsymbol{f}$ is computed from the partial derivatives

$$\frac{\partial f_1}{\partial x}(x, y) = -d + (-ay)\exp(-axy),$$
$$\frac{\partial f_1}{\partial y}(x, y) = (-ax)\exp(-axy),$$
$$\frac{\partial f_2}{\partial x}(x, y) = 1,$$
$$\frac{\partial f_2}{\partial y}(x, y) = -1.$$
$$\boldsymbol{J} = \begin{pmatrix} -d - ay\exp(-axy) & -ax\exp(-axy) \\ 1 & -1 \end{pmatrix}.$$

The function f_2 is linear. This is reflected in the Jacobian matrix, which as constant coefficients on the second row. The evaluation of the Jacobian matrix at steady state $\boldsymbol{x}^* = (0, 0)^t$ is

$$\boldsymbol{J}(\boldsymbol{x}^*) = \begin{pmatrix} -d & 0 \\ 1 & -1 \end{pmatrix}.$$

The evaluation of Jacobian matrix at steady state $\bar{x}_1 = \left(\sqrt{-\frac{\ln d}{a}}, \sqrt{-\frac{\ln d}{a}}\right)^t$ is

$$\boldsymbol{J}(\bar{x}_1) = \begin{pmatrix} -d - a\bar{y}_1\exp(-a\bar{x}_1\bar{y}_1) & -a\bar{x}_1\exp(-a\bar{x}_1\bar{y}_1) \\ 1 & -1 \end{pmatrix}.$$

Here, we use the fact that steady states satisfy the equation $\exp(-axy) = d$ to simplify the exponential terms

$$\boldsymbol{J}(\bar{x}_1) = \begin{pmatrix} -d - a\bar{y}_1 d & -a\bar{x}_1 d \\ 1 & -1 \end{pmatrix}.$$

Replacing y_1 and x_1 by $\sqrt{-\frac{\ln d}{a}}$, we obtain

$$J(\bar{x}_1) = \begin{pmatrix} -d - ay_1 d & -a\bar{x}_1 d \\ 1 & -1 \end{pmatrix},$$

$$= \begin{pmatrix} -d - a\sqrt{-\frac{\ln d}{a}}d & -a\sqrt{-\frac{\ln d}{a}}d \\ 1 & -1 \end{pmatrix},$$

$$= \begin{pmatrix} -d\left(1 + a\sqrt{-\frac{\ln d}{a}}\right) & -d\sqrt{-a^2\frac{\ln d}{a}} \\ 1 & -1 \end{pmatrix},$$

$$= \begin{pmatrix} -d\left(1 + \sqrt{-a\ln d}\right) & -d\sqrt{-a\ln d} \\ 1 & -1 \end{pmatrix}.$$

The same lines of calculations for the steady state $\bar{x}_2$ lead to

$$J(\bar{x}_2) = \begin{pmatrix} -d\left(1 - \sqrt{-a\ln d}\right) & d\sqrt{-a\ln d} \\ 1 & -1 \end{pmatrix}.$$

Solution to exercise 31 TODO

12 Glossary

French	English	Note
dérivable	differentiable	
matrice jacobienne	Jacobian matrix	
ensemble	set	
espace vectoriel	vector space	
sous-espace vectoriel	linear subspace	
valeur propre	eigenvalue	
vecteur propre	eigenvector	
sous-espace propre	eigenspace	
décomposition en valeurs propres	eigenvalue decomposition	
trace	trace	tr
déterminant	determinant	det
application linéaire	linear map	
application	map	
dimension	dimension	
produit scalaire	scalar product	

Index

13 List of exercises