INSA Lyon | Département Biotechnologie & Bioinformatique | 3e année

Refresher course in maths September 17, 2025

Elements of Maths for Biology

with solutions to the exercises

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link to the latest version: Moodle

This document is intended for training and future reference. When important concepts are encountered for the first time, they are highlighted in **bold** next to their definition. We tried to provides examples that are as complete as possible. This means that they are long, you could probably solve them faster. Exercises are important, they can introduce theory or techniques that will be prove useful. Solutions to the exercises can be found at the end of the documents. Examples and exercises (and their solutions!) will be added regularly, so feel free download the most recent version (here).

Contents

[1] Based on lecture notes by Laurent Pujo-Menjouet and Samuel Bernard

1 Functions, maps

A **function** is a relation, often denoted f, that associates an element x of a **domain** I, and at most one element y of the **image** J. The domain I and image J are sets. Usually $I, J \in \mathbb{R}$.

function

domain

image

map

A **map** is a relation that associate *each* element of its domain to exactly one element of its image. Maps and functions are related but slightly different concepts. A function f is a map if it is defined for all elements of its domain I. A map is always a function. The term *map* can also be used when the domain or the image are not numbers (Figure 1).

The **graph** of a function f, denoted $\mathcal{G}(f)$ is the set of all pairs $(x, f(x))$ in the $I \times J$ plane. For real-valued functions, the graph is represented as a curve in the Cartesian plane.

graph

Functions are not numbers. Do not confuse

- f the function
- $f(x)$ the evaluation of f at element x; $f(x)$ is an element of the image (usually a number)
- $\mathcal{G}(f)$ the graph of f

Consequently, do not write

- $f(x)$ is increasing... Instead write f is increasing...
- $f(x)$ is decreasing... Instead write f is decreasing...
- $f(x)$ is continuous... Instead write f continuous...

1.1 Some usual maps

- $f : \mathbb{R} \to \mathbb{R}$, with $x \to k$, $k \in \mathbb{R}$ constant; $x \to x$, **identity map**.

identity map

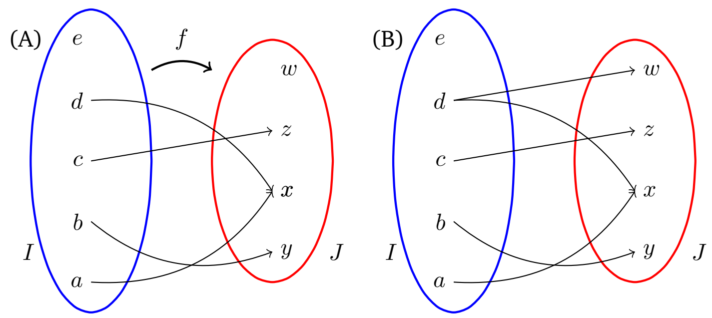
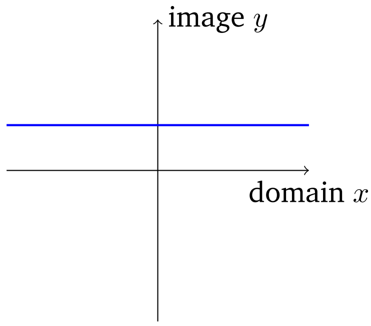
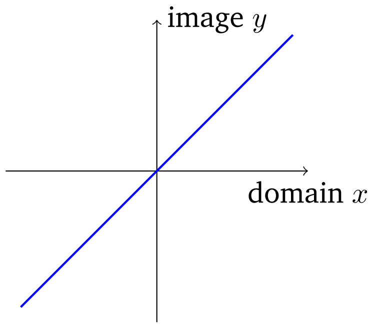


Figure 1. Functions. (A) Function f. (B) Not a function.



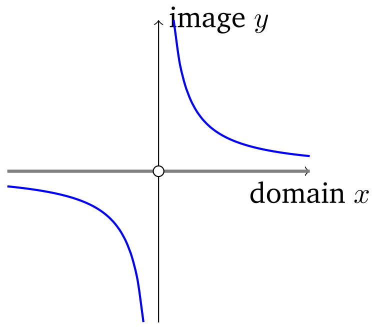
constant map



identity map

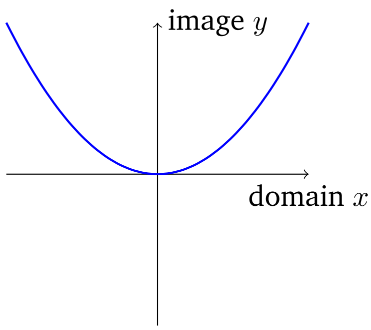
- $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, with $x \to \frac{1}{x}$, **inverse**.

inverse

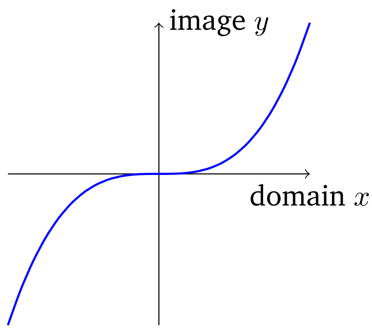


- $f : \mathbb{R} \to \mathbb{R}$, with $x \to x^2$, parabola; $x \to x^3$, **cubic map**.

cubic map



parabola

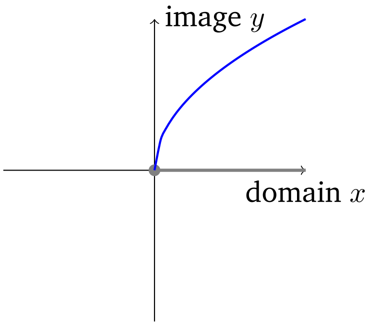


cubic map

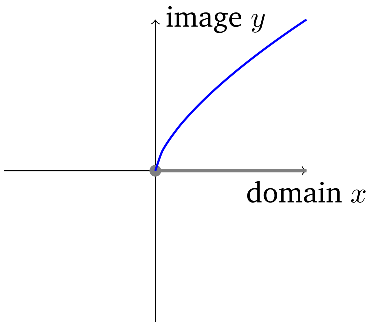
- $f : \mathbb{R}^+ \to \mathbb{R}$, with $x \to \sqrt{x} = x^{\frac{1}{2}}$, **square root**; more generally with $x \to x^{\frac{p}{q}} = \sqrt[q]{x^p}$, **fractional power**.

square root

fractional power

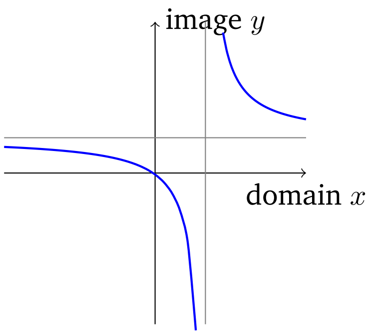


square root



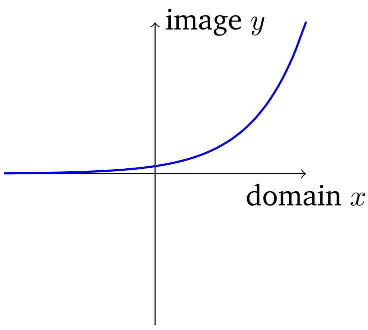
fractional root ($p = 2, q = 3$)

- $f : \mathbb{R}\backslash\{-d/c\} \to \mathbb{R}$, with $x \to \frac{ax+b}{cx+d}$.



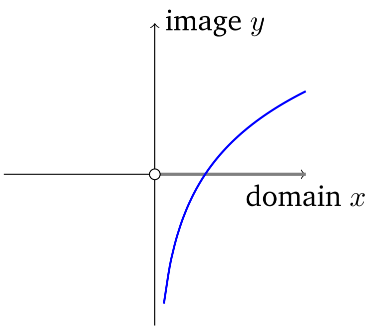
- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \exp(x)$, **exponential.**

exponential



- $f : \mathbb{R}^+\backslash\{0\} \to \mathbb{R}$, with $x \to \ln(x)$, **natural logarithm.**

natural logarithm



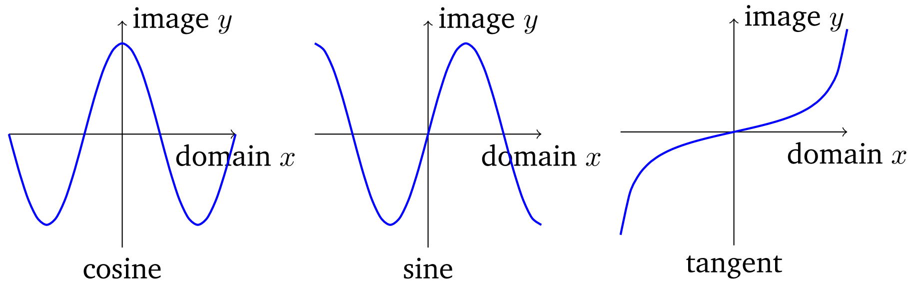
On logarithms: For $a, b > 0$, n positive integer, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^n) = n\ln(a)$, $\ln(a/b) = \ln(a) - \ln(b)$.

- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \cos(x)$, **cosine**; $x \to \sin(x)$, **sine** $x \to \tan(x)$, **tangent**.

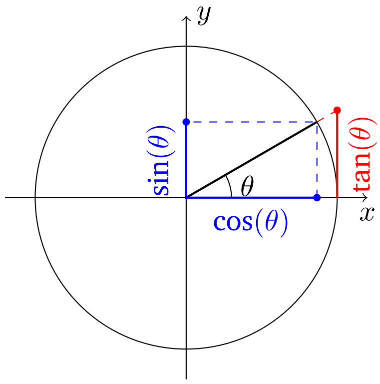
cosine

sine

tangent



A bit more on trigonometric functions. The diagram below shows the relationship between the sine, cosine and tangent, of an angle $\theta \in [0, 2\pi]$.

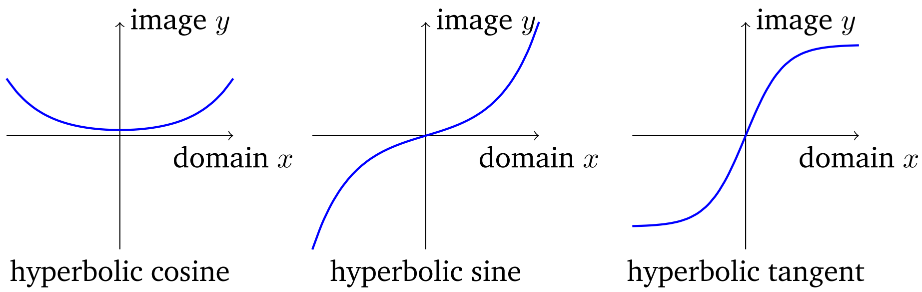


- $f : \mathbb{R} \to \mathbb{R}$, with $x \to \cosh(x) = \frac{1}{2}(e^x + e^{-x})$, **hyperbolic cosine**; $x \to \sinh(x) = \frac{1}{2}(e^x - e^{-x})$, **hyperbolic sine**; $x \to \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, **hyperbolic tangent**.

hyperbolic cosine

hyperbolic sine

hyperbolic tangent



1.2 Exercises on functions

Exercise 1 (Functions). Draw the graph of the function f defined by $f(x) = \frac{x}{K+x}$, for $x \geq 0$, and $K > 0$. *(Solution to exercise 1)*

Exercise 2 (Functions). Among the following maps, which one are increasing? Decreasing?

- $x \to x(x-3)(x-3)$, $x \in \mathbb{R}$.
- $x \to -x^3$, $x \in \mathbb{R}$.

- $x \to \sin(x)$, $x \in \mathbb{R}$.
- $x \to \frac{x^2}{1+x^2}$, $x \geq 0$.

(Solution to exercise 2)

Exercise 3 (Monotonicity).

- Determine whether the function $f(x) = x^3 - 3x$ is increasing, decreasing, or neither on $\mathbb{R}$.
- For $f(x) = e^{-x^2}$, find the intervals where it is increasing and where it is decreasing.
- Does $f(x) = \ln(x)$ increase or decrease on $(0, \infty)$?

(Solution to exercise 3)

Exercise 4 (Boundedness).

- Is the function $f(x) = \sin(x) + \cos(x)$ bounded? If so, find its bounds.
- Decide whether $f(x) = \frac{1}{x}$ is bounded on the intervals: (i) $(0, 1)$ (ii) $(1, \infty)$.
- Check if $f(x) = -5x + 6$ is bounded above, bounded below, or unbounded.

(Solution to exercise 4)

Exercise 5 (Continuity).

- Is
$$f(x) = \frac{x^2 - 1}{x - 1}$$
continuous at $x = 1$?
- For $f(x) = |x|$, determine whether it is continuous everywhere.
- Investigate the continuity of
$$f(x) = \begin{cases} x^2, & x \leq 2, \\ 3x - 2, & x > 2 \end{cases}$$
at $x = 2$.

(Solution to exercise 5)

Exercise 6 (Limits and asymptotic behavior).

- Find
$$\lim_{x \to \infty} \frac{2x + 3}{x + 5}.$$
- Does $f(x) = \tan(x)$ have a horizontal asymptote?
- Determine whether $f(x) = \frac{\sin x}{x}$ has a limit as $x \to \infty$ and as $x \to 0$.

(Solution to exercise 6)

Exercise 7 (Properties of functions).

- Is $f(x) = x^4 + 1$ even, odd, or neither?
- For $f(x) = \frac{1}{1+x^2}$, decide if it is bounded, continuous, and monotone on $[0, \infty)$.
- Consider $f(x) = e^x$. Is it bounded? Is it monotone?

(Solution to exercise 7)

Exercise 8 (Zeroes).

- Find all zeroes of $f(x) = x^2 - 5x + 6$.
- Does the function $f(x) = e^x + 2$ have any zeroes? Justify your answer.
- Determine the real zeroes (if any) of $f(x) = \sin(x) - \frac{1}{2}$.

(Solution to exercise 8)

2 Derivatives

We call the **derivative** of the function $f : I \to J$ ($I, J \subset \mathbb{R}$), at point $a \in I$ the limit, if it exists,

derivative

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The derivative is denoted $f'(a)$. An alternative representation of the limit is obtained by setting $h = x - a$,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If the derivative exists for all elements $a \in I$, we say that **differentiable** on I.

differentiable

- If f is differentiable on I, and $f'(x) > 0$, then f is strictly increasing on I.
- If f is differentiable on I, and $f'(x) < 0$, then f is strictly decreasing on I.

However, if f is strictly increasing, it does not mean that $f'(x) > 0$. For example the function f with $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$. Where the derivative exists, we can define the derivative function $f' : I \to \mathbb{R}$ of f.

The **second derivative** of a function f, denoted f'' is the derivative of f', where defined. If $f''(x)$ exists and $f''(x) > 0$ for all $x \in I$, we say that f is **convex** (U-shaped). If $f'(x) = 0$ and $f''(x) > 0$, the point x is a **minimum**. If $f'(x)$ and $f''(x) < 0$, the point x is a **maximum**. Maxima and minima are **extrema**. If $f''(0) = 0$, the point x is an **inflection point** (Figure 2).

second derivative

convex

minimum

maximum

extrema

inflection point

2.1 List of common derivatives

The derivative has linear properties. If f and g are differentiable on I, and $a \in \mathbb{R}$,

- $(f + g)' = f' + g'$.
- $(af)' = a(f')$.

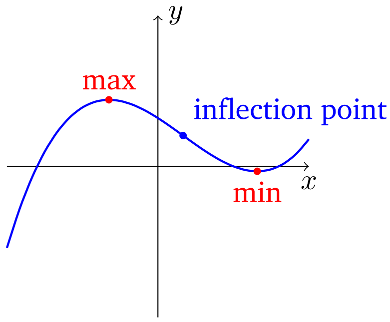


Figure 2. Extrema, inflection points of the polynomial $f(x) = (x + 0.8)(x - 0.5)(x - 0.8)$.

- $(af + g)' = a(f') + g'$.

Let $g : I \to J$ and $f : J \to K$ be two functions. The **composition** of f and g, denoted $f \circ g$, is the function $x \to f(g(x))$, i.e. $f \circ g(x) = f(g(x))$. If f and g are differentiable, the composition $f \circ g$ is also differentiable, and its derivative follows the **rule of composed functions**:

composition

rule of composed functions

$$\big(f \circ g\big)'(x) = f'(g(x))g'(x).$$

Example 1 Let $f : x \to x^2$ and $g : x \to 3x + 1$ be two differentiable functions, with $f'(x) = 2x$ and $g'(x) = 3$. The derivative of the composed function $f \circ g$ at x is

$$f'(g(x))g'(x) = f'(3x + 1)g'(x) = 2(3x + 1) \cdot 3 = 6(3x + 1) = 18x + 6.$$

The derivative could have been obtained by first computing the composed function $f(g(x)) = (3x + 1)^2 = 9x^2 + 6x + 1$, and then taking the derivative.

Example 2 Compute the derivative of $f : x \to \sin(1/x)$. The function f is composed of a sine and an inverse function. To compute the derivative, we decomposed the function f as $f(x) = g(h(x))$ with $g(x) = \sin(x)$ and $h(x) = 1/x$. The derivatives are $g'(x) = \cos(x)$ and $h'(x) = -1/x^2$. Finally the derivative of f is

$$f'(x) = g'(h(x))h'(x) = \cos(1/x)\Big(\frac{-1}{x^2}\Big) = -\frac{\cos(1/x)}{x^2}.$$

Example 3 A function $f : I \to I$ is bijective (**invertible**) on I if there exists a function, denoted f^{-1} and called **inverse** of f, such that the compositions are equal $f \circ f^{-1} = f^{-1} \circ f$, and are equal to the identity map. That is, $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$ for all $x \in I$. If f is differentiable and invertible, what is the derivative of f^{-1}?

invertible

inverse

We apply the derivative to $f(f^{-1})$. Given that $f(f^{-1}(x)) = x$ by definition, we have $\big(f(f^{-1})\big)' = 1$, and

$$\begin{aligned}\big(f(f^{-1})\big)'(x) &= f'(f^{-1}(x))(f^{-1})'(x),\\ &= 1,\\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}.\end{aligned}$$

Take for instance $f(x) = x^2$ on $x \in (0, 1]$. The inverse of f is $f^{-1}(x) = \sqrt{x}$. The derivative of f is $f'(x) = 2x$ and the derivative

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(\sqrt{x})}.$$

Example 4 Compute a derivative when the independent variable is in the exponent. To compute the derivative of $f : x \to 2^x$, we need to re-express the function in terms of the natural base e. To do that, we use to properties of the natural logarithm

- For any positive expression y, $y = e^{\ln(y)}$ (ln is the inverse of the exponential function).
- $\ln(a^b) = b\ln(a)$.

Then $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$. The derivative is $\ln(2)e^{x\ln(2)}$. Re-writing in term of base 2, we obtain $f'(x) = \ln(2)2^x$.

Function	Derivative	Note
x^a	ax^{a-1}	$a \in \mathbb{R}$
$\frac{1}{x}$	$\frac{-1}{x^2}$	
$x^{\frac{1}{2}}$	$\frac{1}{2x^{\frac{1}{2}}}$	
$\ln(x)$	$\frac{1}{x}$	
e^x	e^x	
$\cosh(x)$	$\sinh(x)$	
$\sinh(x)$	$\cosh(x)$	
$\cos(x)$	$-\sin(x)$	
$\sin(x)$	$\cos(x)$	
$\tan(x)$	$1 + \tan^2(x) = \frac{1}{\cos^2(x)}$	
$\frac{u(x)}{v(x)}$	$\frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}$	
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	

2.2 Exercises on derivatives

Exercise 9 (Derivatives). Compute the derivatives of the following functions

- $f_1 : x \to \sqrt{\cos x}$.
- $f_2 : x \to \sin(3x + 2)$.
- $f_3 : x \to e^{\cos x}$.
- $f_4 : x \to \ln(\sqrt{x})$.
- $f_5 : x \to 2^{\ln x}$.

(Solution to exercise 9)

Exercise 10 (Extrema).

- Find the local extrema of $f(x) = x^3 - 3x^2 + 4$.

- Does the function $f(x) = \cos(x)$ attain a maximum or minimum on $[0, 2\pi]$? If yes, where?
- Determine whether $f(x) = \ln(x)$ has a local maximum or minimum on $(0, \infty)$.

(Solution to exercise 10)

Exercise 11 (Inflection points).

- Find the inflection points of $f(x) = x^3 - 6x^2 + 9x$.
- Does the function $f(x) = e^x$ have any inflection points?
- Determine whether $f(x) = \tan(x)$ has inflection points on its domain.

(Solution to exercise 11)

3 Taylor series and truncated expansions

If a function f is infinitely differentiable (i.e. the k-th derivative function $f^{(k)}$ is continuous for any integer $k \geq 0$), the Taylor series of f at point a is the series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

A function is **analytic** on an open interval I if and only if its Taylor series converges pointwise to the value of the function. Polynomials, exponential and trigonometric function are analytic over all real points. The square root function is not analytic.

analytic

The partial sums of the series (or truncated expansion) can be used to approximate a function, and to evaluate it numerically. The k-th order expansion of a function f is the polynomial

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Truncated expansions are used in implementations of common mathematical functions in computer programs.

Example 5 The Taylor series of the sine function at point $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n \sin(x)}{dx^n} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

The 3rd-order expansion is the cubic polynomial

$$x - \frac{x^3}{3!}.$$

How good this cubic polynomial approximation to the original sine function? The error is the remainder of the terms of the Taylor series

$$\left| -\frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \right|.$$

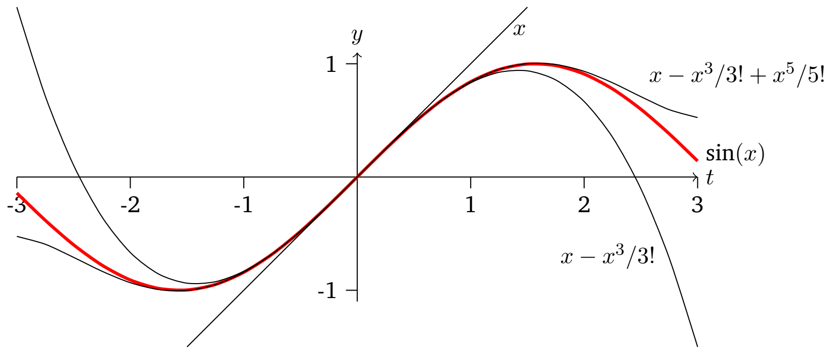


Figure 3. First, third and fifth order expansion of $f(x) = \sin(x)$.

For $|x| < 1$, the error is bounded by $|x|^7/7!$. Given that $7! = 5040$, the error is less than $1/5040 \approx 0.0002$. Truncated expansions are never good approximations when $|x|$ becomes large, because polynomials are not bounded, but the approximation can be quite good over small intervals around the point at which the Taylor series is computed.

It is important to note that even if a function has a Taylor series, the series may not converge to the function.

Example 6 Let the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0. \end{cases} .$$

Around 0, all the derivatives exist and are equal to 0. The Taylor series of f at point $x = 0$ is 0, but the function itself is different from the identically zero function.

Exercise 12 (Taylor expansion around zero).

- Find the Taylor expansion of $f(x) = e^x$ around $x = 0$ up to order 4.
- Expand $f(x) = \sin(x)$ around $x = 0$ up to order 5.
- Compute the Taylor polynomial of order 3 of $f(x) = \ln(1 + x)$ around $x = 0$.

(Solution to exercise 12)

Exercise 13 (Taylor expansion at non-zero points).

- Find the Taylor expansion of $f(x) = \cos(x)$ around $x = \pi/2$ up to order 2.
- Compute the Taylor polynomial of order 3 of $f(x) = \sqrt{1 + x}$ around $x = 1$.
- Determine the Taylor expansion of $f(x) = \ln(x)$ around $x = 2$ up to order 2.

(Solution to exercise 13)

Exercise 14 (Applications of Taylor expansion).

- Use the Taylor expansion of $\ln(1 + x)$ around 0 to approximate $\ln(1.1)$ with an error less than 10^{-3}.

- Approximate $\sin(0.1)$ using the Taylor polynomial of order 3 at $x = 0$. Estimate the remainder term.
- Use the second-order Taylor polynomial of $f(x) = e^x$ at $x = 0$ to approximate $e^{0.5}$.

(Solution to exercise 14)

Exercise 15 (Error estimates of Taylor expansion).

- Write the remainder term for the Taylor expansion of $f(x) = e^x$ at $x = 0$, order n, and give a bound for the error when approximating e with $n = 5$.
- Show that the remainder of the Taylor expansion of $\sin(x)$ at 0 of odd order n is bounded by $\frac{|x|^{n+1}}{(n+1)!}$.
- Estimate the error when approximating $\ln(1.5)$ using the Taylor polynomial of $\ln(1+x)$ at $x = 0$, order 3.

(Solution to exercise 15)

3.1 Expansion of a function of two variables

For functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}$, the Taylor expansion at point $(x_0, y_0)^t \in \mathbb{R}^2$ is

$$
\begin{aligned}
f(x,y) = f(x_0,y_0) &+ \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0) \\
&+ \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial x^2}(x-x_0)^2 \\
&+ \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial y^2}(y-y_0)^2 \\
&+ \frac{2}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}(x-x_0)(y-y_0) + \dots
\end{aligned}
$$

Example 7 The second-order truncated expansion of the function $f(x,y) = ye^{-x}$ at $(0,0)^t$ is

$$y - xy.$$

Exercise 16 (Taylor expansion with two variables).

- Find the second-order Taylor polynomial of
$$f(x,y) = x^2 + y^2$$
around $(0,0)$.
- Expand $f(x,y) = e^{x+y}$ around $(0,0)$ up to order 2.
- Compute the Taylor polynomial of order 3 of
$$f(x,y) = \sin(x)\cos(y)$$
around $(0,0)$.

(Solution to exercise 16)

Exercise 17 (Taylor expansion with two variables at non-zero points).

- Find the second-order Taylor expansion of $f(x, y) = \ln(1+x+y)$ around $(0, 0)$.
- Compute the Taylor polynomial of order 2 of $f(x, y) = \sqrt{1 + x^2 + y^2}$ around $(0, 0)$.
- Determine the second-order Taylor expansion of $f(x, y) = e^x \sin(y)$ around $(1, 0)$.

(Solution to exercise 17)

Exercise 18 (Applications of Taylor expansion with two variables).

- Use the second-order Taylor polynomial of $f(x, y) = e^{x+y}$ at $(0, 0)$ to approximate $f(0.1, 0.1)$.
- Approximate $\sqrt{1.02^2 + 0.01^2}$ using the second-order Taylor expansion of $f(x, y) = \sqrt{x^2 + y^2}$ around $(1, 0)$.
- Use the third-order Taylor expansion of $f(x, y) = \ln(1+x+y)$ around $(0, 0)$ to approximate $\ln(1.05 + 0.02)$.

(Solution to exercise 18)

3.2 Expansion of a function from $\mathbb{R}^2 \to \mathbb{R}^2$

Functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ have the form

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}.$$

To compute the Taylor series, we need to compute the Taylor series of each function f_1 and f_2. For a first-order expansion at $(x_0, y_0)^t$, we obtain the expansion

$$\begin{pmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f_1(x_0,y_0)}{\partial y}(y - y_0) \\ \frac{\partial f_2(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f_2(x_0,y_0)}{\partial y}(y - y_0) \end{pmatrix}$$

Using vector notation

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix},$$

$$D\boldsymbol{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix},$$

we can write a first-order (or linear) approximation of a function $\boldsymbol{f}$ more compactly with

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{a}) + D\boldsymbol{f}(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}).$$

The approximation is valid only in a neighbourhood of the point $\boldsymbol{a}$.

Exercise 19 (Taylor expansion of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$). Let

$$F(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x^2 + y \\ e^x \cos(y) \end{pmatrix}.$$

- Compute the second-order Taylor expansion of $F(x,y)$ around $(0,0)$.
- Write the expansion in the form

$$F(x,y) \approx F(0,0) + J_F(0,0) \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} (x,y) H_{f_1}(0,0) \begin{pmatrix} x \\ y \end{pmatrix} \\ (x,y) H_{f_2}(0,0) \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix},$$

  where $J_F(0,0)$ is the Jacobian matrix and $H_{f_i}(0,0)$ are the Hessian matrices of the components.
- Use the expansion to approximate

$$F(0.1, 0.05).$$

(Solution to exercise 19)

4 Integrals and primitives

4.1 Primitives

Let $f : I \to \mathbb{R}$, $(I \subset \mathbb{R})$. A **primitive** F of f on I is a differentiable map such that $F'(x) = f(x)$, $x \in I$. primitive

Example 8 If $f(x) = x^2$, we look for $F(x)$ such that $F'(x) = x^2$. Take $F(x) = x^3/3$, then $F'(x) = 3x^2/3 = x^2$. It also work for $F(x) = x^3/3 + C$, for any constant $C \in \mathbb{R}$.

We denote the primitive as $F(x) = \int f(x)dx$. Primitives are unique up to a constant: if F_1 and F_2 are primitives of f, then $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$, which implies that $F_1'(x) - F_2'(x) = 0$. Therefore the difference between F_1 and F_2, $G : x \to F_1(x) - F_2(x)$ has a zero derivative: $G'(x) = F_1'(x) - F_2'(x) = 0$. This means that G is a constant.

Primitives have linear properties. Let F be a primitive of f, G be a primitive of g and $a, b \in \mathbb{R}$. Then

- $F + G$ is a primitive of $f + g$, or

$$\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx.$$

- aF is a primitive of af, or

$$\int af(x)dx = a \int f(x)dx.$$

Function	Primitive	Note
0	C	$C \in \mathbb{R}$
a	$ax + C$	$a \in \mathbb{R}$
x^a	$\frac{x^{a+1}}{a+1}$	$a \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$	$(a = -1)$, $x \neq 0$
e^x	$e^x + C$	
$\cos(x)$	$\sin(x) + C$	
$\sin(x)$	$-\cos(x) + C$	
$f'(g(x))g'(x)$	$f(g(x)) + C$	

Exercise 20 (Primitives).

- Compute $\int \frac{1}{\sqrt{3x+5}} dx$.
- Find a primitive of $f(x) = \cos(2x)$.
- Compute $\int (2x)\, e^{x^2}\, dx$.
- Find $\int \frac{1}{x \ln x}\, dx$.
- Compute $\int x \sin x\, dx$.
- Evaluate $\int \frac{1}{x^2 - 1}\, dx$ (write the result using partial fractions).
- Find $\int \frac{2x+3}{x^2+3x+2}\, dx$.
- Compute $\int \sin^2 x\, dx$.
- Compute $\int \frac{dx}{(x+1)^2}$.
- Compute $\int \frac{1}{\sqrt{1-x^2}}\, dx$.
- Find $\int \frac{x^2}{(x+1)^2}\, dx$.

(Solution to exercise 20)

4.2 Integrals

Let $f : [a, b] \to \mathbb{R}$, and F a primitive of f on $[a, b]$. Then the **definite integral** is the value $F(b) - F(a)$, and is denoted

definite integral

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b.$$

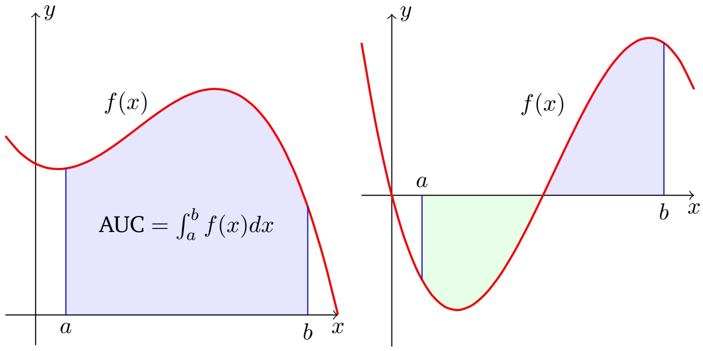
The most important interpretation of the integral is the **area under the curve** (AUC).

area under the curve

If $f(x) \geq 0$, $x \in [a, b]$, then

$$\int_a^b f(x)dx$$

is the area under delimited by $f(x)$, x-axis and the axes $x = a$ and $x = b$. Negative values integrate as negative areas.



If f is a rate (unit in `something per time`), and x is `time`, then the integral of f has unit `something`. This applies to speed (integral: displacement), production or degradation (integral: concentration), etc.

The integral has the following properties

- Linearity. The integral of a sum is the sum of the integrals.

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

- Negative intervals.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

- Midpoint rule.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

  Notice that this works even if c is not between a and b.

- The expression

$$\int_a^x f(t)dt = F(x) - F(a) = F_a(x)$$

  is a function of x. Therefore, for any integrable function f, we have

$$\left(\int_a^x f(x)dx\right)' = f(x).$$

  This is the **fundamental theorem of calculus**.

- **Integration by parts**

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx.$$

fundamental theorem of calculus

Integration by parts

Integration by part is excessively useful for computing integral beyond simple functions.

Exercise 21 (Integrals). Compute the integral

$$\int_0^1 xe^x dx.$$

(Solution to exercise 21)

Exercise 22 (Integrals). Compute the integral

$$\int_0^\pi \cos xe^x dx.$$

(Solution to exercise 22)

Exercise 23 (Integrals). Compute the integral

$$\int_0^1 \sinh xe^x dx.$$

(Solution to exercise 23)

5 Differential equations in 1D

An **ordinary differential equation** (ODE) of **order** n is a relation (i.e. an equation) between a real variable $t \in I \subset \mathbb{R}$, an unknown function $x : t \to x(t)$ and its derivatives $x', x'', x''', ..., x^{(n)}$ at point t defined by

ordinary differential equation

order

$$F\big(t, x, x', x'', ..., x^{(n)}\big) = 0,$$

where F *depends* on $x^{(n)}$. In general, $x(t)$ takes values in $\mathbb{R}^N$, i.e. it is a vector. We say that the equation is a **scalar differential equation** if $N = 1$. (The expression $x^{(i)}$ stands for the i-th derivative, not the i-th power.)

scalar differential equation

The **normal form** of a differential equation of order n is

normal form

$$x^{(n)} = f\big(t, x, x', x'', ..., x^{(n-1)}\big).$$

A differential equation is **autonomous** if it does not depend on t, i.e. F has the form

autonomous

$$F\big(x, x', x'', ..., x^{(n)}\big) = 0,$$

A **linear differential equation** is a differential equation for which F is a linear function in $x, x', ...x^{(n)}$. It can be expressed as

linear differential equation

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + ...a_1(t)x' + a_0(t)x = g(t),$$

where the coefficients $a_j(t)$ may depend on t, and x and its derivatives $x^{(i)}$ appear only as monomials of degree 1 (that is, linearly). If all coefficients are constants, including g, the linear differential equation is autonomous.

Exercise 24 (Types of differential equations). For the following differential equations, give the order n, and determine whether they are autonomous, linear, and whether they are expressed under their normal form.

1. $x - t + 4tx' = 0.$
2. $\left(x''\right)^2 - 2x'tx = 0.$
3. $x^{(3)} + \sin(x') = -5x.$
4. $x^{(4)} - xx' = 0.$
5. $3x'' - 4x' + 6x = 2.$
6. $\ln x' + 3tx = \sqrt{x}.$

(Solution to exercise 24)

A **solution or integral of a differential equation** of order n for t in an interval $I \subset \mathbb{R}$, is a map $x : I -> \mathbb{R}$ that is n times differentiable for all $t \in I$ and satisfies the differential equation.

solution or integral of a differential equation

A **integral curve (or chronic)** is the set of points $(t, x(t))$ for $t \in I$. If $x(t) \in \mathbb{R}^N$, the integral curve is in $\mathbb{R}^{N+1}$. A **trajectory or orbit** is the set of points $x(t)$ for $t \in I$. This is a set in $\mathbb{R}^N$. The space that contains the trajectories is called the **phase space**. The set of all trajectories is called the **phase portrait**.

integral curve (or chronic)

trajectory or orbit

phase space

phase portrait

It is always possible to write an differential equation of order n as a differential equation of order 1, by defining extra variables for the higher-order derivatives.

Example 9 We consider the differential equation $a(t)x'' + b(t)x' + c(t)x = d(t)$. This is a equation of order 2. To reduce it to order 1, let $z_1 = x$ and $z_2 = x'$. Then $x'' = z_2'$ and $z_1' = z_2$. The second-order differential equation can be re-expressed as two first order equations:

$$z_1' = z_2, \quad a(t)z_2' + b(t)z_2 + c(t)z_1 = d(t).$$

Often, the system of first order equations can be re-expressed in normal form, by isolating the variables z_1' and z_2',

$$z_1' = z_2, \text{ and } z_2' = \frac{d(t) - b(t)z_2 - c(t)z}{a(t)}.$$

(We assume that $a(t) \neq 0$.) In general, for the differential equation of order n $F(t, x, x', ..., x^{(n)}) = 0$, with $x : I \to \mathbb{R}^m$, we make a change of variables: $z_1 = x, z_2 = x', ..., z_i = x^{(i-1)}$ until $z_n = x^{(n-1)}$. Each variable $z_i(t) \in \mathbb{R}^m$, so the new vector $z = \left(z_1, z_2, ..., z_n\right)^t$ is in $\mathbb{R}^{mn}$. With the change of variables, the differential equation now reads

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= z_3, \\ &\dots, \\ z_i' &= z_{i+1}, \\ &\dots, \\ F(t, z_1, z_2, \dots, z_n, z_n') &= 0. \end{aligned}$$

Tips on Ordinary differential equations

- The most frequently used differential equations are order 1, and they usually are represented in their normal form: $x' = f(x)$ for autonomous equations, and $x' = f(t, x)$ for non-autonomous equations
- For a scalar, autonomous differential equation $x' = f(x)$ with $x(t) \in \mathbb{R}$, the trajectories are monotonous: if x is a solution, then x is either increasing, decreasing, or constant.

5.1 Finding solutions of differential equations

We consider a **first order scalar differential equation**,

first order scalar differential equation

$$a(t)x' + b(t)x = d(t),$$

$t \in I$ and $a(t) \neq 0$ on I, $a(t)$ and $b(t)$ continuous on I. If $d(t) = 0$, we call the equation homogeneous, and

$$a(t)x' + b(t)x = 0.$$

First order scalar ODE. Homogeneous case, first method. Write the equation in normal form,

$$x' = -\frac{b(t)}{a(t)}x.$$

The solution x is either the constant $x = 0$, or $x(t) \neq 0$ for all $t \in I$. We know that because of uniqueness of solutions, which implies that trajectories cannot cross. If $x = 0$ we are done, so we can assume that $x(t) \neq 0$. Dividing the equation by x

$$\frac{x'}{x} = -\frac{b(t)}{a(t)}.$$

The terms on both sides are functions of t. We can compute their primitives

$$\int \frac{x'}{x} dt = -\int \frac{b(t)}{a(t)} dt.$$

The integrand of the left-hand side is x'/x. This is a very common form called **log-derivative** and admits the primitive $\ln |x|$. The right-hand side does not necessarily

log-derivative

have a close form, and we leave it as it is. With the integration constant, we obtain implicit solution for x

$$\ln|x| = -\int \frac{b(t)}{a(t)}dt + K.$$

We would like an explicit solution x,

$$|x| = e^{-\int \frac{b(t)}{a(t)}dt+K}.$$

Therefore,

$$x = \pm e^K e^{-\int \frac{b(t)}{a(t)}dt}.$$

Notice that t is a variable of integration, and does not exist outside the integral, and can be replaced by any other variable. To have x as as function of t, we must define the bounds of the integral. When the domain of definition of t is the interval $I = [t_0, t_1], t_0 < t_1$, the solution $x(t)$ is obtained by integrating from t_0 to t,

$$x = \pm e^K e^{-\int_{t_0}^{t} \frac{b(u)}{a(u)}du},$$

where we have replaced the variable of integration by u. Then the definite integral is a function of t. The constant K is determined by the **initial condition** on x at t_0, $x(t_0) = x_0 \in \mathbb{R}$,

initial condition

$$x(t_0) = \pm e^K e^{-\int_{t_0}^{t_0} \frac{b(u)}{a(u)}du} = \pm e^K = x_0.$$

The constant $K = \operatorname{sign} x_0 \ln x_0$. The fixed bound at t_0 is arbitrary, we could have chosen any other time of reference. However, it is very common to consider differential equations for which we know the value of the solution at the initial time t_0. The problem of solving a differential equation with a given initial condition is called an **initial value problem** (IVP) or a **Cauchy problem**.

initial value problem

Cauchy problem

First order scalar ODE. Homogeneous case, second method. Assume that $a(t) \neq 0$, and write the equation as

$$x' + \frac{b(t)}{a(t)}x = 0.$$

Multiply the equation by the term $e^{\int \frac{b(t)}{a(t)}dt}$,

$$x'e^{\int \frac{b(t)}{a(t)}dt} + \frac{b(t)}{a(t)}xe^{\int \frac{b(t)}{a(t)}dt} = 0.$$

Notice that the first term has the form $x'f$ and the second term the form xf', where the term f is

$$f = e^{\int \frac{b(t)}{a(t)}dt}.$$

The left-hand-side of the resulting equation, $x'f + xf'$, is the derivative of the product xf, so the differential equation can be integrated,

$$xe^{\int \frac{b(t)}{a(t)}dt} = K,$$

$$x = Ke^{-\int \frac{b(t)}{a(t)}dt}.$$

The constant K is determined by a condition set on the solution, as in the first method.

Exercise 25 (Differential equations). Solve the equation

$$2x' + 6x = 0, \quad x(0) = 1.$$

(Solution to exercise 25)

Exercise 26 (Differential equations). Solve the equation

$$x' + \frac{1}{t}x = 0, \quad x(1) = 1.$$

(Solution to exercise 26)

First order scalar ODE. Heterogeneous case. We now consider the more general differential equation

$$a(t)x' + b(t)x = d(t).$$

Using the strategy from the second method for the homogeneous case above, we divide the equation by $a(t)$, again assuming that $a(t) \neq 0$, and then multiply by $e^{\int \frac{b(t)}{a(t)}dt}$. The resulting equation now has a non-zero right-hand-side,

$$x'e^{\int \frac{b(t)}{a(t)}dt} + \frac{b(t)}{a(t)}xe^{\int \frac{b(t)}{a(t)}dt} = \frac{d(t)}{a(t)}e^{\int \frac{b(t)}{a(t)}dt}.$$

Nevertheless, the left-hand-side is still of the form $x'f + xf'$, and the right-hand-side only depends on t. By integrating both sides, we obtain

$$xe^{\int \frac{b(t)}{a(t)}dt} = \int \frac{d(t)}{a(t)}e^{\int \frac{b(t)}{a(t)}dt}dt + K.$$

The solution for x is then

$$x = \Big(\int \frac{d(t)}{a(t)}e^{\int \frac{b(t)}{a(t)}dt}dt + K\Big)e^{-\int \frac{b(t)}{a(t)}dt}.$$

Example 10 We consider the differential equation

$$x' + 3x = 1 + \sin(t), \quad x(0) = x_0 > 0.$$

The coefficients $a(t) = 1, b(t) = 3, d(t) = 1 + \sin(t)$. The term $e^{\int \frac{b(t)}{a(t)}dt} = e^{\int 3dt} = e^{3t}$. The general solution is

$$\begin{aligned} x(t) &= \Big(\int (1 + \sin(t))e^{3t}dt + K\Big)e^{-3t}, \\ &= \frac{1}{3} + \frac{1}{10}\big(\sin(t) - \cos(t)\big) + Ke^{-3t}. \end{aligned}$$

At $t = 0$, the equation $x(0) = \frac{1}{3} + \frac{1}{10}(0 - 1) + K = x_0$ solve in K as $K = x_0 - \frac{7}{30}$. The solution $x(t)$ is

$$x(t) = \frac{1}{3} + \frac{1}{10}\big(3\sin(t) - \cos(t)\big) + \Big(x_0 - \frac{7}{30}\Big)e^{-3t}.$$

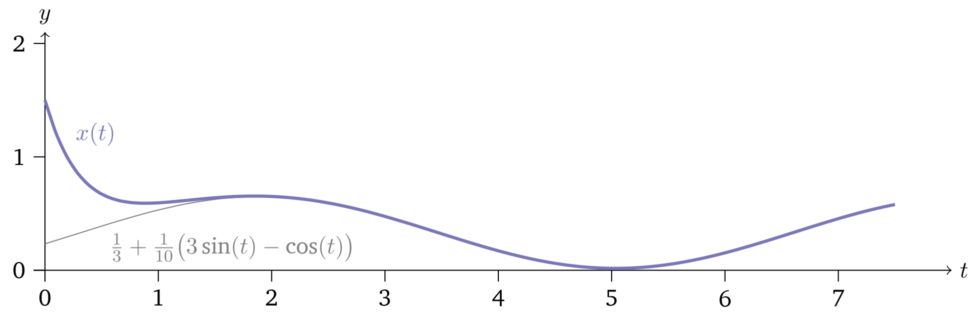


Figure 4. Solution of the initial value problem for the heterogeneous differential equation $x' + 3x = 1 + \sin(x)$.

We now consider a **nonlinear differential equation** of **Bernoulli** type. Bernoulli equations are of the form

nonlinear differential equation

Bernoulli

$$x' + P(t)x + Q(t)x^r = 0, \quad t \in I \subset \mathbb{R},$$

with continuous functions P, Q, and $r \in \mathbb{R}$. There is no general method to solve nonlinear differential equations, but is can be done is particular cases. If $r = 0$ or $r = 1$, the equation is linear, and we already know how to solve it. Suppose r different from 0 or 1. We will look for positive solutions $x(t) > 0$ on $t \in I$. Dividing by x^r, we get

$$\begin{aligned} x'x^{-r} + P(t)xx^{-r} + Q(t)x^r x^{-r} &= 0, \\ x'x^{-r} + P(t)xx^{-r} + Q(t) &= 0. \end{aligned}$$

We now set an auxiliary variable $u = x^{1-r}$, ($x = u^{1/(1-r)}$). Then

$$u' = (1-r)x^{1-r-1}x' = (1-r)x^{-r}x',$$

and, substituting in the differential equation,

$$\begin{aligned} \frac{1}{1-r}u' + P(t)u + Q(t) &= 0, \\ u' + (1-r)P(t)u + (1-r)Q(t) &= 0. \end{aligned}$$

We know how to solve this equation; this is a linear equation of the form

$$a(t)u' + b(t)u = d(t),$$

with $a(t) = 1, b(t) = (1-r)P(t), d(t) = -(1-r)Q(t)$.

Example 11 Verhulst equation (logistic equation)

$$x' = \mu x\Big(1 - \frac{x}{K}\Big).$$

We first rewrite the equation in Bernoulli form,

$$x' - \mu x + \mu\frac{x^2}{K} = 0,$$

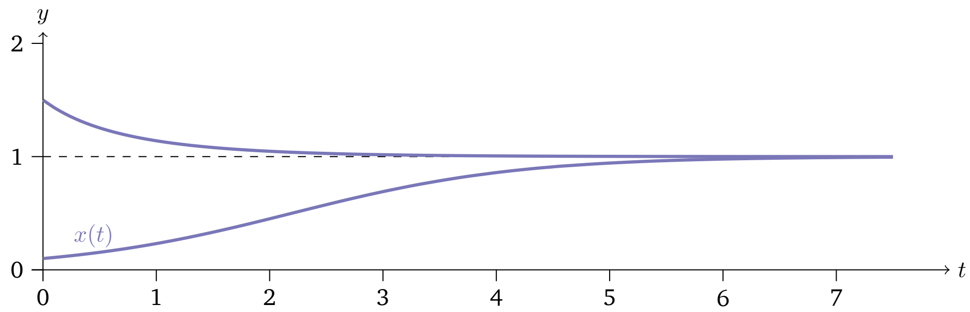


Figure 5. Solution of the initial value problem for the Verhulst equation $x' = \mu x(1 - x/K)$, for $x(0) = 0.1$ and $x(0) = 1.5$. Verhulst equation is a type of Bernoulli equation, and can be solved analytically.

with $P(t) = -1, Q(t) = \mu/K, r = 2$. The auxiliary equation reads

$$u' + (1-2)(-1)\mu u + (1-2)\frac{\mu}{K} = 0.u' + u - \frac{\mu}{K} \qquad = 0.$$

This is a scalar linear equation with $a(t) = 1, b(t) = \mu, d(t) = \mu/K$.. The general solution is

$$\begin{aligned} u(t) &= \Big(\int \frac{d(t)}{a(t)} e^{\int b(t)/a(t)dt} dt + C\Big) e^{-\int b(t)/a(t)dt}, \\ \int b(t)/a(t)dt &= \int \mu dt = \mu t, \\ u(t) &= \Big(\int \frac{\mu}{K} e^{\mu t} dt + C\Big) e^{-\mu t}, \\ &= \Big(\frac{\mu}{K}\frac{e^{\mu t}}{\mu} + C\Big) e^{-\mu t}, \\ &= \frac{1}{K} + Ce^{-\mu t}. \end{aligned}$$

The initial condition $u(0) = x_0^{1-r} = x_0^{-1}$,

$$\begin{aligned} u(0) &= \frac{1}{K} + C = \frac{1}{x_0}, \\ C &= \frac{1}{x_0} - \frac{1}{K}. \end{aligned}$$

The solution to the original Verhulst equation is $x = u^{-1}$,

$$\begin{aligned} x(t) &= \frac{1}{\frac{1}{K} + (\frac{1}{x_0} - \frac{1}{K})e^{-\mu t}} \\ &= \frac{x_0 K}{x_0 + (K - x_0)e^{-\mu t}}. \end{aligned}$$

6 Complex numbers

A complex number is a number that can be expressed in the form $a + ib$, where a and b are real numbers, and the symbol i is called **imaginary unit**. The imaginary

imaginary unit

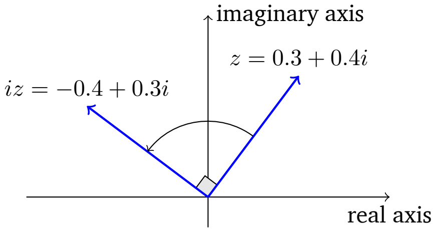


Figure 6. Rotation in the complex plane. Multiplication by i is a 90 degree counterclockwise rotation.

unit satisfies the equation $i^2 = -1$. Because no real number satisfies this equation, this number is called *imaginary*.

For the complex number $z = a + ib$, a is called the **real part** and b is called the **imaginary part**. The real part of z is denoted $\Re(z)$ (`\Re` in LaTeX) or just $\mathrm{Re}(z)$. The imaginary part of z denoted $\Im(z)$ (`\Im` in LaTeX) or just $\mathrm{Im}(z)$. The set of all complex numbers is denoted $\mathbb{C}$ (`\mathbb{C}` in LaTeX).

real part

imaginary part

We need complex numbers for solving polynomial equations. The fundamental theorem of algebra asserts that a polynomial equation of with real or complex coefficients has complex solutions. These polynomial equations arise when trying to compute the eigenvalues of matrices, something we need to do to solve linear differential equations for instance.

Arithmetic rules that apply on real numbers also apply on complex numbers, by using the rule $i^2 = -1$: addition, subtraction, multiplication and division are associative, commutative and distributive.

Let $u = a + ib$ and $v = c + id$ two complex numbers, with real coefficients a, b, c, d. Then

- $u + v = a + ib + c + id = (a + c) + i(b + d)$.
- $uv = (a + ib)(c + id) = ac + iad + ibc + i^2bd = ac - bd + i(ad + bc)$.
- $\frac{1}{v} = \frac{1}{c+id} = \frac{c-id}{(c-id)(c+id)} = \frac{c-id}{c^2+d^2} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}$.
- $u = v$ if and only if $a = c$ and $b = d$.

  It follows from the rule on i that
- $\frac{1}{i} = -i$. (Proof: $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$.)

Multiplying by the imaginary unit i is equivalent to a counterclockwise rotation by $\pi/2$ (Figure 6)

$$ui = (a + ib)i = ia + i^2b = -b + ia.$$

Let $z = a + ib$ a complex number with real a and b. The **conjugate** of z, denoted $\bar{z}$, is $a - ib$. The conjugate of the conjugate of z is z (*reflection*, Figure 6). The **modulus** of z, denoted $|z|$ is $\sqrt{z\bar{z}}$. The product $z\bar{z} = (a+ib)(a-ib) = a^2+b^2+i(-ab+ab) = a^2+b^2$. The modulus is the complex version of the absolute value, for if z (i.e. $b = 0$), $|z| = \sqrt{a^2} = |a|$. It is always a real, positive number, and $|z| = 0$ if and only if $z = 0$.

conjugate

modulus

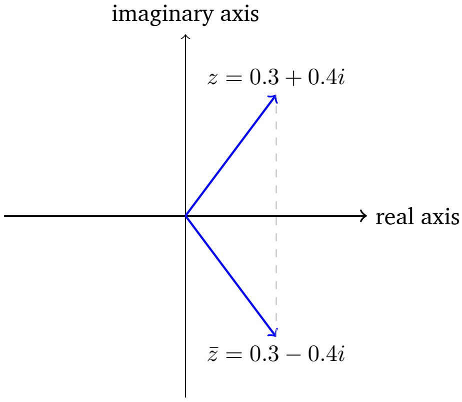


Figure 7. The conjugate of a complex z is the complex $\bar{z}$ obtained by a reflection with respect to the real axis.

The modulus also has the property of being the *length* of the complex number z, if a and b are the sides of a rectangular triangle, then $|z|$ is the hypotenuse.

When simplifying a ratio involving a complex v at the denominator, it is important to convert it to a real number by multiplying the ratio by $\bar{v}/\bar{v}$. For instance, if $v \neq 0$,

$$\frac{u}{v} = \frac{u\bar{v}}{u\bar{v}} = \frac{u\bar{v}}{|v|^2}.$$

The denominator $|v|^2$ is always a positive real number.

By allowing complex values, nonlinear functions of real numbers like exponential, logarithmic and trigonometric functions can have their domain extended to all real and complex numbers. The most useful extension is the exponential function. Recall that the exponential function e^x, where $e \approx 2.71828$ is Euler's constant, satisfies the relation $e^{x+y} = e^x e^y$. This remains true for complex numbers. The **Euler's formula** relates the exponential of a imaginary number with trigonometric functions. For a real number y,

Euler's formula

$$e^{iy} = \cos(y) + i\sin(y).$$

Therefore, for any complex number $z = a + ib$, the exponential

$$e^z = e^{a+ib} = e^a e^{ib} = e^a\big(\cos(b) + i\sin(b)\big).$$

Tips on complex numbers

- If x is real, ix is pure imaginary. If y is imaginary, iy is real.
- $|i| = 1$. For any real θ, $|e^{i\theta}| = 1$.
- $|z_1 z_2| = |z_1||z_2|$.
- In particular, $|iz| = |i||z| = |z|$. (Multiplying by i is a rotation in the complex plane, it does not change the modulus.)

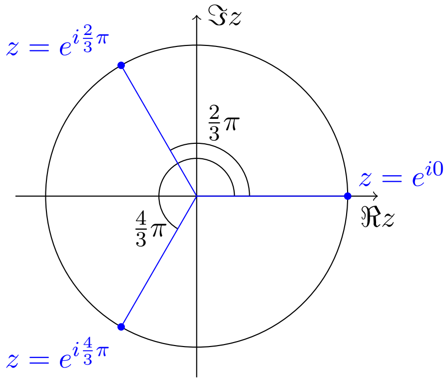


Figure 8. The roots of the polynomial $z^3 - 1$.

6.1 Roots of a complex number

For complex numbers, the equation $z^n = 1$ has n solutions. They are called the **roots of unity**. For $n = 2$, we have the well-known roots $z = \pm 1$, which are real. What are the roots of $z^3 = 1$? To find them, we express z in polar coordinates: $z = re^{i\theta}$. Then

roots of unity

$$z^3 = (re^{i\theta})^3 = r^3 e^{i3\theta} = 1.$$

The equation implies that z has modulus 1, so $r = 1$. The remaining term $e^{i3\theta} = 1$ implies that 3θ is a multiple of 2π because $e^{i\omega} = 1$ if and only if $\omega = 2k\pi$, for some integer k. Therefore $\theta = \frac{2}{3}k\pi$, for $k = 0, 1, 2, \ldots$. How many distinct points do we have? Clearly, $k = 3$ is equivalent to $k = 0$: $e^{i\frac{2}{3}3\pi} = e^{i2\pi} = ei0$. In the same way $k = 4$ is equivalent to $k = 1$, and so on. Therefore, there are exactly three distinct solutions for θ: $0, \frac{2}{3}\pi, \frac{4}{3}\pi$ (Figure 8).

6.2 Exercises on complex numbers

Exercise 27 (Complex numbers). Let $z = 2 + 3i$. Compute the values of $\bar{z}$, $|z|$, $|\bar{z}|$ (compare with $|z|$), z^2, $\Re(\bar{z})$, $\Im(\bar{z})$, $\frac{z+\bar{z}}{2}$, $\frac{z-\bar{z}}{2}$, $-z$, iz. *(Solution to exercise 27)*

Exercise 28 (Complex numbers). Any complex number can be represented in **polar form**: $z = r(\cos(\theta) + i\sin(\theta))$.

polar form

- Show that $|z| = r$
- Show that $z = re^{i\theta}$
- Conclude that for any complex number z, $|z| = 1$ if and only if z can be expressed as $z = e^{i\theta}$ for a real θ.

(Solution to exercise 28)

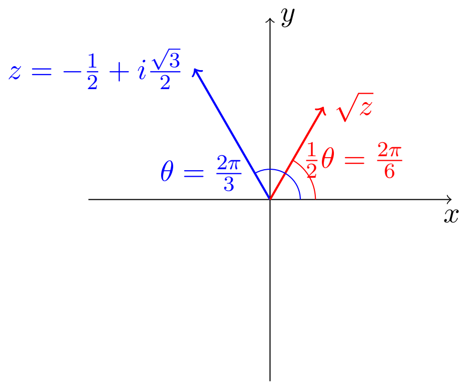
Exercise 29 (Euler formula). Using Euler's formula, show that $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a + b)$. *(Use the property that $e^{ia+ib} = e^{ia}e^{ib}$ and apply Euler's Formula)*. *(Solution to exercise 29)*

Exercise 30 (Euler formula). Show Euler's identity: $e^{i\pi} = -1$. *(Solution to exercise 30)*

Exercise 31 (Roots of unity). What are the roots of the equation $z^6 = 1$? *(Solution to exercise 31)*

Exercise 32 (Complex numbers). For a complex z, find necessary and sufficient conditions for e^{zt}, $t > 0$, to converge to 0. *(Solution to exercise 32)*

Exercise 33 (Complex numbers). Let the complex number $z = a+ib$ with real a and b. Compute $\sqrt{z}$ (that is, express $s = \sqrt{z}$ as $s = \alpha + i\beta$, with real α and β) *(Solution to exercise 33)*



7 Matrices in dimension 2

7.1 Eigenvalues of a 2×2 matrix

A 2×2 matrix A is an array with 2 rows and 2 columns:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Usually, the **coefficients** a, b, c, d are real numbers. The **identity** matrix is the matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

coefficients

identity

The **determinant** of A, denoted $\det A$ or $|A|$ is the number $ad - bc$. The **trace** of A, denoted $\operatorname{tr} A$, is the sum of the main **diagonal** of A: $a + d$.

determinant

trace

diagonal

The **characteristic polynomial** of A is the second order polynomial in λ obtained by computing the determinant of the matrix $A - \lambda I$ (Figure 9),

characteristic polynomial

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = ad - bc - \lambda(a + d) + \lambda^2.$$

The characteristic polynomial $p_A(\lambda)$ of A can be expressed in terms of its determinant and trace:

$$p_A(\lambda) = \det A - \operatorname{tr} A\lambda + \lambda^2.$$

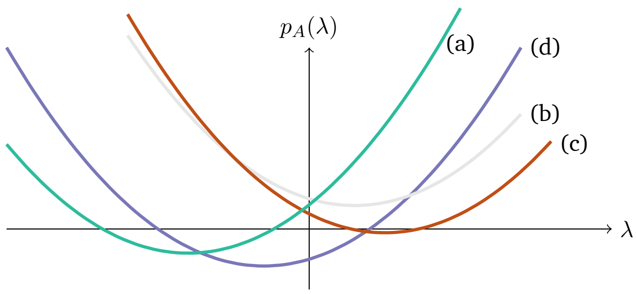


Figure 9. Graph of the characteristic polynomial. (*a, green*) polynomial with two negative roots, $p(\lambda) = 0.08+0.8\lambda+\lambda^2$. (*b, gray*) polynomial with two complex roots, $p(\lambda) = 0.1-0.3\lambda+\lambda^2$. (*c, red*) polynomial with two positive roots, $p(\lambda) = 0.05 + 0.5\lambda + \lambda^2$. (*d, purple*) polynomial with a negative and a positive root, $p(\lambda) = -0.1 - 0.3\lambda + \lambda^2$.

The **eigenvalues** of A are the roots of the characteristic polynomial. By the fundamental theorem of algebra, we know that the characteristic polynomial has exactly two roots, counting multiple roots. These roots can be real, or complex. The eigenvalues of A are calculated using the quadratic formula:

eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right).$$

From this formula, we can classify the eigenvalues of A. Let

$$\Delta = (\operatorname{tr} A)^2 - 4\det A$$

the **discriminant** of the quadratic formula. The two eigenvalues of A are real if and only if $\Delta \geq 0$, i.e. $\operatorname{tr} A)^2 \geq 4\det A$ Then we have the following properties (Figure 10):

discriminant

1. $\Delta < 0$, complex eigenvalues
   - The two eigenvalues are complex conjugate: $\lambda_1 = \bar{\lambda}_2$
   - Their real part $\Re(\lambda) = \frac{1}{2}\operatorname{tr} A$.
2. $\Delta = 0$, there is a single root of multiplicity 2: $\lambda = \frac{1}{2}\operatorname{tr} A$.
3. $\Delta > 0, \det A > 0$, real, distinct eigenvalues of the same sign.
   - $\operatorname{tr} A > 0$ and $\det A > 0$. Then $\lambda_{1,2}$ are distinct and positive.
   - $\operatorname{tr} A < 0$ and $\det A > 0$. Then $\lambda_{1,2}$ are distinct and negative.
4. $\det A < 0$, real distinct eigenvalues of opposite sign.
   - $\lambda_1 < 0 < \lambda_2$.
5. $\det A = 0$ one of the eigenvalue is zero, the other eigenvalue is $\operatorname{tr} A$.

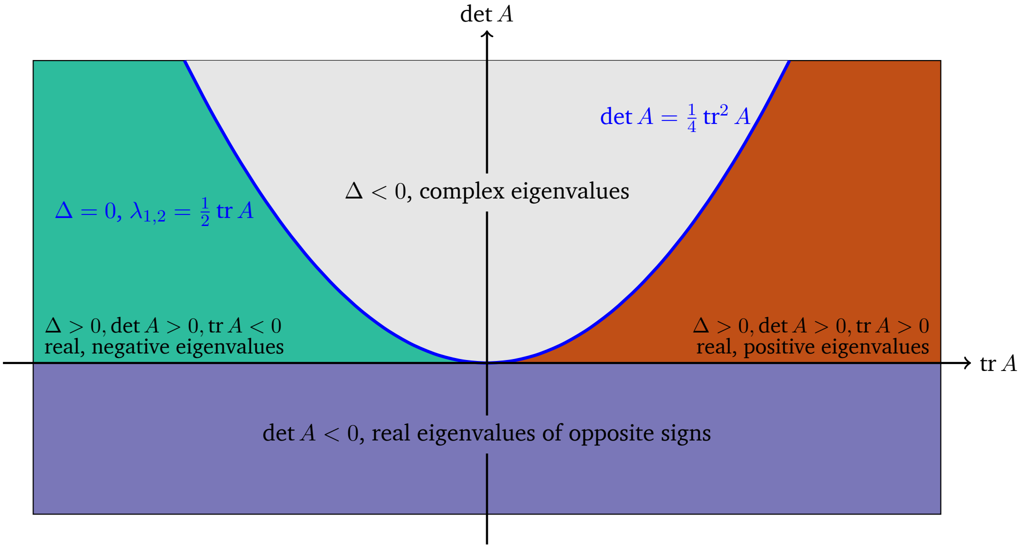


Figure 10. Properties of the eigenvalues of a 2×2 matrix A with respect to $\det A$ and $\mathrm{tr}\,A$.

7.1.1 Exercises on eigenvalues

Exercise 34 (Eigenvalues). Properties of the eigenvalues of 2×2 matrices. For each 2×2 matrix, compute the determinant, the trace, and the discriminant, and determine whether the eigenvalues are real, complex, distinct, and the sign (negative, positive, or zero) of the real parts.

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 2 \\ 1/2 & 2 \end{pmatrix}.$$

(Solution to exercise 34)

7.2 Matrix-vector operations

A matrix defines a linear transformation between vector spaces. Given a vector x, the product Ax is vector composed of linear combinations of the coefficients of x. For a matrix 2×2, the vector x must be a vector of size 2, and the product Ax is a vector of size two. If $x = (x_1, x_2)^t$ (the ^{t} stands for the transpose, because x must be a column vector), and $A = [a_{ij}]_{i=1,2,\, j=1,2}$, then

$$Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

Successive linear transformations can be accomplished by applying several matrices. Given two matrices A, B, the matrix product $C = AB$ is also a matrix. The matrix C is the linear transformation that first applies B, then A. Matrix product is *not* commutative is general: $AB \neq BA$. *(If B means 'put socks on' and A means 'put* *shoes on', then BA does not have the expected result!)* The product of two matrices $A = [a_{ij}]_{i=1,2,\, j=1,2}$ and $B = [b_{ij}]_{i=1,2,\, j=1,2}$ is

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

The **sum of two matrices** $A + B$ is performed element-wise: $A + B = [a_{ij} + b_{ij}]_{i=1,2,\, j=1,2}$. The **sum of two vectors** is defined similarly. Addition is commutative. Matrix operations are associative and distributive.

sum of two matrices

sum of two vectors

$$\begin{aligned} A + B &= B + A, \\ A(B + C) &= AB + BC, \\ A(BC) &= (AB)C. \end{aligned}$$

Matrices and vectors can be multiplied by a scalar value (real or complex). **Multiplication by a scalar** is associative, distributive, and commutative. The result of the multiplication by a scalar is to multiply each coefficient of the matrix or vector by the scalar. For example, if λ, μ are scalars,

Multiplication by a scalar

$$\begin{aligned} \lambda A &= A(\lambda I) = A\lambda, \\ \lambda(A + B) &= \lambda A + \lambda B, \\ (\lambda A)B &= \lambda(AB) = A(\lambda B), \\ (\mu + \lambda)A &= \mu A + \lambda A, \\ \mu(\lambda A) &= (\mu\lambda)A, \ldots \end{aligned}$$

The product between two column vectors is not defined, because the sizes do not match. However, we can define the **scalar product** between two column vectors x, y in the same way matrix product is defined:

scalar product

$$x^t y \equiv x_1 y_1 + x_2 y_2.$$

If the vectors are complex-valued, we need also to conjugate the transposed vector x^t. The conjugate-transpose is called the **adjoint** and is denoted *. Thus, if x is complex-valued, the adjoint x^* is the row vector $(\bar{x}_1, \bar{x}_2)$. The scalar product for complex-valued vectors is denoted $x^* y$. Since this notation also works for real-valued vector, we will use it most of the time.

adjoint

Two vectors are **orthogonal** if their scalar product is 0. In the plane, this means that they are oriented at 90 degree apart. Orthogonal vectors are super important because they can be used to build orthogonal bases that are necessary for solving all sorts of *linear problems*.

orthogonal

7.2.1 Exercises on Matrix-vector and matrix-matrix operations

Exercise 35 (Matrix-vector product). Compute matrix-vector product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

What is the transformation given by this matrix? *(Solution to exercise 35)*

Exercise 36 (Matrix-matrix product). Compute the matrix-matrix product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Can you tell what transformation this is? *(Solution to exercise 36)*

Exercise 37 (Matrix-matrix product). Now compute the product of the same matrices, but in the inverse order

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Compare with the solution found in the previous exercise. What is this transformation? *(Solution to exercise 37)*

Exercise 38 (Matrix construction). Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns the vector $(ax_1, bx_2)^t$. *(Solution to exercise 38)*

Exercise 39 (Matrix construction). Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns the vector $(x_2, x_1)^t$. *(Solution to exercise 39)*

Exercise 40 (Matrix construction). Find the matrix that takes a vector $x = (x_1, x_2)^t$ and returns the vector $(x_2, 0)^t$. *(Solution to exercise 40)*

Exercise 41 (Power of a matrix). Compute the successive powers $A, A^2, A^3, \ldots$, for a diagonal matrix A:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

(Solution to exercise 41)

Exercise 42 (Scalar product). Compute the scalar product x^*y between $x = (1 + 2i, 1 - i)^t$ and $y = (0.5 - i, -0.5)^t$. *(Solution to exercise 42)*

Exercise 43 (Scalar product). Now compute the scalar product y^*x and compare with the result with the previous exercise. *(Solution to exercise 43)*

Exercise 44 (Scalar product). Compute the scalar product between $z = (z_1, z_2)^t$ and itself, if z is a complex-valued vector. What can you say about the result? *(Solution to exercise 44)*

Tips on eigenvalues Some matrices have special shapes that make it easier to compute the determinant, and the eigenvalues. These are called eigenvalue-revealing shapes.

- Diagonal matrices have their eigenvalues on the diagonal.
- **Triangular matrices**, i.e. matrices that have zeros above (lower-triangular matrix) or below (upper-triangular matrix) the main diagonal have also their eigenvalues on the diagonal.

Triangular matrices

- A matrix with a row or a column of zeros has its determinant equal to zero. This implies that one of its eigenvalues is 0.

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8 Eigenvalue decomposition

In many applications, it is useful to decompose a matrix into a form that makes it easier to operate complex operations on. For instance, we might want to compute the powers of a matrix A: A^2, A^3, A^4. Multiplying matrices are computationally intensive, especially when the size of the matrix becomes large. The **power of a matrix** is $A^k = AA...A$, k times. The zeroth power is the identity matrix: $A^0 = I$.

power of a matrix

The **inverse** of a matrix A, denoted by A^{-1} is the unique matrix such that $AA^{-1} = A^{-1}A = I$. The notation is self-consistent with the positive powers of A. The inverse of a matrix does not always exist. A matrix is **invertible** if and only if its determinant is not 0. If A and B are invertible, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

inverse

invertible

The **eigenvalue decomposition** is a decomposition of the form $A = XDX^{-1}$, where D is a diagonal matrix, and X is an invertible matrix. If there exists such a decomposition for A, then computing powers of A becomes easy:

eigenvalue decomposition

$$\begin{aligned} A^k = (XDX^{-1})^k &= XDX^{-1}\, XDX^{-1} \,...XDX^{-1}, \\ &= XD(X^{-1}X)D(X^{-1}X)D...(X^{-1}X)DX^{-1}, \\ &= XD^kX^{-1}. \end{aligned}$$

The eigenvalue decomposition does not always exists, because it is not always possible to find an invertible matrix X. When it exists, though, the columns of the matrix X is composed of the eigenvectors of A. When A is a 2×2 matrix, it is enough to find 2 linearly independent eigenvectors x and y for the matrix

$$X = \left(\begin{array}{c|c} x_1 & y_1 \\ x_2 & y_2 \end{array}\right)$$

to be invertible.

8.1 Eigenvectors

The **eigenvectors** of a matrix A are the *nonzero* vectors x such that for an eigenvalue λ of A,

eigenvectors

$$Ax = \lambda x.$$

If x is an eigenvector, so is any αx for any scalar value α. If there are two linearly independent eigenvectors x and y associated to an eigenvalue, $\alpha x + \beta y$ is also an eigenvector. There is at least one eigenvector for each distinct eigenvalue, but there may be more than one when the eigenvalue is repeated.

Example 12 Distinct, real eigenvalues The matrix

$$A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$$

is upper-triangular; this is one of the eigenvalue-revealing shapes. The eigenvalues are -1 and 1. These are distinct eigenvalues, so each eigenvalue possesses a single eigenvector. The eigenvector x associated to $\lambda_1 = -1$ is found by solving the eigensystem

$$Ax = (-1)x.$$

The unknown quantity x appears on both sides of the equation. We can find a simpler form by noting that multiplying a vector by the identity matrix is neutral: $(-1)x = (-1)Ix$. The eigenproblem becomes

$$\begin{aligned} Ax &= (-1)Ix, \\ Ax - (-1)Ix &= 0, \\ \big(A - (-1)I\big)x &= 0, \end{aligned}$$

that is, the eigenvector is a nonzero solution of the linear system $\big(A - \lambda I\big)x = 0$. In general, if a matrix B is invertible, the only solution to $Bx = 0$ is $x = 0$ (the vector of zeroes). But, by construction, $A - \lambda I$ cannot be invertible if λ is an eigenvalue: its determinant is exactly the characteristic polynomial evaluated at one of its roots, so it is zero. This is why the eigensystem has nonzero solutions. Now, because $A - \lambda I$ is not invertible, this means that a least one of its rows is a linear combination of the others. For 2×2 matrices, this implies that the two rows are co-linear, or redundant. For our example, the eigensystem reads

$$\begin{aligned} \begin{pmatrix} -1-(-1) & -2 \\ 0 & 1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

we immediately see that the two rows $(0, -2)$ and $(0, 2)$ are co-linear, with a factor -1. This leads to an underdetermined system: $0x_1 + -2x_2 = 0$. The solution is $x_2 = 0$ and we can take x_1 to be any value, save 0. We choose $x = (1, 0)^t$.

For the eigenvalue $\lambda_2 = +1$, the eigensystem reads:

$$\begin{aligned} \begin{pmatrix} -1-(+1) & -2 \\ 0 & 1-(+1) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Again, the second row $(0, 0)$ can be neglected, and the solution is $-2y_1 - 2y_2 = 0$, or $y_1 = -y_2$. It is customary to choose an eigenvector with norm 1. The **norm** of a complex-valued vector $y = (y_1, y_2)^t$ is

norm

$$||y|| = \sqrt{y^* y} = \sqrt{\bar{y}_1 y_1 + \bar{y}_2 y_2} = \sqrt{|y_1|^2 + |y_2|^2}.$$

Here, the eigenvector is $y = (y_1, -y_1)^t$, so $||y|| = \sqrt{|y_1|^2 + |-y_1|^2} = \sqrt{2}\sqrt{|y_1|^2} = \sqrt{2}|y_1|$. Taking $||y|| = 1$ solves $|y_1| = 1/\sqrt{2}$. This means that we could take a negative or a complex value for y_1, as long as the $|y_1| = 1/\sqrt{2}$. Going for simplicity, we take $y_1 = 1/\sqrt{2}$.

Example 13 Complex eigenvalues

The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is *not* diagonal, so we have to compute the eigenvalues by hand. The trace of A is zero, the determinant is $0 - (1)(-1) = 1$, and the discriminant is -4. A negative discriminant implies complex eigenvalues,

$$\lambda_{1,2} = \frac{1}{2}\big(0 \pm \sqrt{-4}\big) = \pm i.$$

For the eigenvalue $\lambda_1 = +i$, the eigensystem reads:

$$\begin{pmatrix} -(+i) & -1 \\ 1 & -(+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two rows $(-i, 1)$ and $(1, -i)$ should be co-linear, but this is not obvious with the complex coefficients. Multiplying the first row by i gives $i(-i, -1) = (-i^2, -i) = (-(-1), -i) = (1, -i)$, the second row, OK. Having confirmed that the system is indeed underdetermined, we can week a solution to $-ix_1 - x_2 = 0$. Solving for $x_2 = -ix_1$, we obtain the eigenvector $x = (x_1, -ix_1)^t$. Normalization of x imposes

$$||x|| = \sqrt{|x_1|^2 + |-ix_1|^2} = \sqrt{|x_1|^2 + |x_1|^2} = \sqrt{2}|x_1| = 1.$$

As in the previous example, we can choose $x_1 = 1/\sqrt{2}$.

The second eigenvectors, associated $\lambda_2 = -i$, solves the eigensystem

$$\begin{pmatrix} -(-i) & -1 \\ 1 & -(-i) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row yields $iy_1 - y_2 = 0$, so $y = (y_1, iy_2)^t$. A normalized eigenvector can be $y = (1/\sqrt{2}, i/\sqrt{2})^t$. We could also have chosen $y = (i/\sqrt{2}, -1/\sqrt{2})^t$.

Example 14 Repeated eigenvalues 1

The matrix

$$\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$$

is lower-triangular, with repeated eigenvalues on the diagonal, $\lambda_{1,2} = -1$. The eigenvectors associated with -1 satisfy the eigenproblem

$$\begin{pmatrix} -1 - (-1) & 0 \\ 2 & -1 - (-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row vanishes, and the second row means that $x_1 = 0$, leaving for instance $x_2 = 1$, and $x = (0, 1)^t$. There are no other linearly independent eigenvectors. This is not always the case, repeated eigenvalues can have more than one independent eigenvector, as in the next example.

Example 15 Repeated eigenvalues 2

The matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is diagonal, with repeated eigenvalues on the diagonal, $\lambda_{1,2} = -1$. The eigenvectors associated with -1 satisfy the eigenproblem

$$\begin{pmatrix} -1-(-1) & 0 \\ 0 & -1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now, the two rows vanished, leaving no condition at all on x_1 and x_2. This means that all the vectors are eigenvectors! How many linearly independent eigenvectors can we find? Vectors of size 2 live in a vector space of dimension 2; we can find at most 2 linearly independent vectors. We can choose for instance the canonical basis: $x = (1, 0)^t$ and $y = (0, 1)^t$.

Tips on eigenvalue decomposition

- A 2×2 matrix (or any square matrix) admits an eigenvalue decomposition if all the eigenvalues are distinct. For 2×2 matrices, eigenvalues are distinct if and only if the discriminant $\Delta \neq 0$.
- If the matrix has a repeated eigenvalue, it will admit an eigenvalue decomposition if the number of (linearly independent) eigenvectors is equal to the number of times the eigenvalue is repeated. The number of eigenvectors is called geometric multiplicity, and the number of repeats is called algebraic multiplicity.
- The eigenproblem should be underdetermined; you should always be able to eliminate at least one row by linear combination. If you cannot, this means that there is a error, possibly an incorrect eigenvalue, or a arithmetic mistake in computing $A - \lambda I$.
- Because eigenvalues are in general complex, the eigenvectors will also be complex.
- The eigenvector matrix X needs to be inverted. When the eigenvectors can be chosen so that they are orthogonal and normalized, the inverse $X^{-1} = X^*$ (i.e. the conjugate transpose of X). Symmetric matrices have orthogonal eigenvalues, so this class of matrices are especially easy to diagonalise.
- Eigenvalue decomposition and matrix inversion are two different concepts. A matrix can be invertible without admitting an eigenvalue decomposition, and vice versa.

- When a matrix does not admit an eigenvalue decomposition, it still can be triangularised. One such triangularisation is the Jordan decomposition: $A = P(D+S)P^{-1}$, where P is invertible, D is the diagonal matrix of eigenvalues, and S is a **nilpotent** matrix, i.e. a nonzero matrix such that $S^k = 0$ for $k \geq k_0 > 1$. nilpotent

8.2 Exercises on eigenvalues decomposition

Exercise 45 (Eigenvalue decomposition). Find, if there is any, an eigenvalue decomposition of

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

To compute X^{-1}, you can use the fact that because A is real and symmetrical, the eigenvectors are orthogonal, meaning that $X^{-1} = X^t$, if the eigenvectors are normalized. *(Solution to exercise 45)*

Exercise 46 (Eigenvalue decomposition). Find, if there is any, an eigenvalue decomposition of

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

(Solution to exercise 46)

9 Linearisation of functions $\mathbb{R}^2 \to \mathbb{R}^2$

Nonlinear systems of ordinary differential equations are used to describe the **dynamics** (evolution in time) of concentration of biochemical species, population densities in ecological systems, or the electrophysiology of neurons. dynamics

Two-dimensional systems are described by a set of two **ordinary differential equations**, or ODEs, ordinary differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2),$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2).$$

The variables x_1, x_2 are functions of time: $x_1(t)$, $x_2(t)$, and f_1, f_2 are the derivatives. We define the two-dimensional vectors $\boldsymbol{x} = (x_1, x_2)^t$ (here we will use **bold** for vectors), and $\boldsymbol{f} = (f_1, f_2)^t$. The ODEs can now be represented in vector format,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}).$$

Here we assume that there exists a point in the 2D plane $\bar{\boldsymbol{x}}$ such that the derivative $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$. This point is called a **steady state** because the derivatives are all zeros; the steady state is therefore a solution to the system of ODE. steady state

We are interested in how $\boldsymbol{f}$ is behaving around the steady state. To do that we linearize the function $\boldsymbol{f}$ at the steady state. **Linearisation** is a first-order expansion. Linearisation

For a function from $\mathbb{R}^2 \to \mathbb{R}^2$, a first-order expansion around a point $\boldsymbol{x}_0$ is

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \boldsymbol{Df}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)$$

When expanding around a steady state, the constant term $\boldsymbol{f}(\bar{\boldsymbol{x}}) = 0$. In the second term, $\boldsymbol{Df}$ is a 2×2 matrix, called the Jacobian matrix, and often denoted $\boldsymbol{J}$. The **Jacobian matrix** for the function $\boldsymbol{f}$ is defined as

Jacobian matrix

$$\boldsymbol{J} = \boldsymbol{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

When evaluated at a steady state, the Jacobian matrix can provide information on the dynamics of the nonlinear ODE system. More precisely, the eigenvalues of the Jacobian matrix can determine whether the steady state is stable (attracts solutions) or is unstable. **Linearisation around a steady state means computing the Jacobian matrix at the steady state.**

Example 16 Linearisation around a steady state

The Lotka-Volterra equations is a classical ODE system mathematical biology. The equations reads

$$\frac{dx}{dt} = ax - xy,$$
$$\frac{dy}{dt} = xy - by,$$

for a, b positive constants. The solution vector is $\boldsymbol{x} = (x, y)^t$ and the derivatives are $f_1(x, y) = ax - xy$ and $f_2 = xy - by$. We first look for steady states

$$f_1 = ax - xy = 0, \quad f_2 = xy - by.$$

If x and y are not zero, we have $x = b$ and $y = a$. If $x = 0$, the second equation implies $y = 0$. If $y = 0$, the first equation implies $x = 0$. Therefore there are two steady states, $\bar{\boldsymbol{x}} = (b, a)^t$ and $\hat{\boldsymbol{x}} = (0, 0)^t$.

We have the following derivatives

$$\frac{\partial f_1}{\partial x}(x, y) = a - y,$$
$$\frac{\partial f_1}{\partial y}(x, y) = -x,$$
$$\frac{\partial f_2}{\partial x}(x, y) = y,$$
$$\frac{\partial f_2}{\partial y}(x, y) = x - b,$$

The Jacobian matrix is

$$\boldsymbol{J} = \begin{pmatrix} a - y & -x \\ y & x - b \end{pmatrix}.$$

Evaluated at the steady state $\bar{\boldsymbol{x}} = (b, a)^t$ and $\hat{\boldsymbol{x}} = (0, 0)^t$, the Jacobian matrices are

$$\boldsymbol{J}(\bar{\boldsymbol{x}}) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}, \quad \boldsymbol{J}(\hat{\boldsymbol{x}}) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}.$$

9.1 Exercises on linearisation

Exercise 47 (Linearisation). Let the function $\boldsymbol{f} = (f_1, f_2)^t$, with

$$f_1(x, y) = -dx + x\exp(-axy), \quad f_2(x, y) = x - y,$$

$d < 1$, a, d positive. Find the steady states (by solving the equations $f_1 = 0, f_2 = 0$). Compute the Jacobian matrix, and evaluate the Jacobian matrix at each steady state. *(Solution to exercise 47)*

Exercise 48 (Jacobian matrices). Compute the Jacobian matrices of each of the following functions of (x, y). All parameters are constants. You do not need to compute the equilibria, just the matrices.

- van der Pol oscillator

$$f_1(x, y) = \mu\big((1 - x^2)y - x\big), \; f_2(x, y) = y.$$

- Two-compartment pharmacokinetics

$$f_1(x, y) = a - k_{12}x + k_{21}y - k_1 x, \; f_2(x, y) = k_{12}x - k_{21}y.$$

- SI epidemiological model

$$f_1(x, y) = -\beta xy, \; f_2(x, y) = \beta xy - \gamma y.$$

(Solution to exercise 48)

10 Solution of systems of linear differential equations in dimension 2

Linear differential equations have linear derivative parts, which can be represented in matrix-vector format

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t),$$

for a vector $\boldsymbol{x}$ square matrix $\boldsymbol{A}$. For initial conditions $\boldsymbol{x}(t) = \boldsymbol{x}_0$, the **solution of the linear system of ODEs** is

solution of the linear system of ODEs

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

If we have at our disposal an eigenvalue decomposition of $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$, the **exponential of the matrix** is

exponential of the matrix

$$
\begin{aligned}
e^{\boldsymbol{A}t} &= \boldsymbol{X} e^{\boldsymbol{D}t} \boldsymbol{X}^{-1}, \\
&= \boldsymbol{X} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \boldsymbol{X}^{-1}.
\end{aligned}
$$

Therefore, the long-time behaviour of the exponential is controlled by the eigenvalues $\lambda_{1,2}$.

Example 17 Solution of a linear system of ODEs

Consider the linear system of ODEs given by the Lotka-Volterra model linearized at its nonzero steady state $\bar{\boldsymbol{x}} = (b, a)^t$ is

$$
\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{1}
$$

This system approximates the nonlinear version near the steady state. In this linear system, variables (x, y) are deviations from the steady state; their solutions are "centred" around 0. To solve this linear system, we will diagonalise the matrix

$$
A = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}.
$$

The goal is to go slowly through every step once for this system. In general it is not necessary to solve the system completely by hand; knowledge of the eigenvalues is often sufficient in many applications.

We have $\det A = 0 - a(-b) = ab > 0$, $\operatorname{tr} A = 0$ and $\Delta = 0 - 4ab = -4ab < 0$. The eigenvalues are therefore complex conjugates: $\lambda_{1,2} = \pm i\sqrt{ab}$. Distinct eigenvalues means that A is diagonalisable. The eigenvector associated to $\lambda_1 = i\sqrt{ab}$ is given by the system

$$
\left(\begin{array}{cc|c} -i\sqrt{ab} & -b & 0 \\ a & -i\sqrt{ab} & 0 \end{array} \right)
$$

We have from the first row $-i\sqrt{ab}x = by$. Letting $x = b$ and $y = -i\sqrt{ab}$, we obtain the non-normalized eigenvector $\tilde{x}_1 = (b, -i\sqrt{ab})^t$. Normalization is done by dividing by

$$
||\tilde{x}|| = \sqrt{b^2 + (-i\sqrt{ab})^2} = \sqrt{b^2 + ab},
$$

to obtain the first eigenvector

$$
x = \begin{pmatrix} \frac{b}{\sqrt{b^2+ab}} \\ \frac{-i\sqrt{ab}}{\sqrt{b^2+ab}} \end{pmatrix} = \begin{pmatrix} \frac{b}{\sqrt{b}\sqrt{b+a}} \\ \frac{-i\sqrt{a}\sqrt{b}}{\sqrt{b}\sqrt{b+a}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{b}}{\sqrt{b+a}} \\ \frac{-i\sqrt{a}}{\sqrt{b+a}} \end{pmatrix}.
$$

The second eigenvector is computed the same way (watch out for the slightly different signs!). The eigenproblem for the eigenvalue $\lambda = -i\sqrt{ab}$ is

$$
\left(\begin{array}{cc|c} +i\sqrt{ab} & -b & 0 \\ a & +i\sqrt{ab} & 0 \end{array} \right)
$$

Given that the only change is $-i \to +i$, the second eigenvector is

$$x_2 = \begin{pmatrix} \frac{\sqrt{b}}{\sqrt{b+a}} \\ \frac{i\sqrt{a}}{\sqrt{b+a}} \end{pmatrix}.$$

The solution to the linear ODE is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \boldsymbol{X} e^{\boldsymbol{D}t} \boldsymbol{X}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

with

$$\boldsymbol{X} = \frac{1}{\sqrt{b+a}} \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} +i\sqrt{ab} & 0 \\ 0 & -i\sqrt{ab} \end{pmatrix}$$

The **inverse of a** 2×2 **matrix** with coefficients a, b, c, d is

inverse of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is conditional to $\det = ad - bc \neq 0$, of course. With this formula, the inverse of $\boldsymbol{X}$ is

$$\boldsymbol{X}^{-1} = \frac{1}{\sqrt{b+a}} \frac{1}{\det \boldsymbol{X}} \begin{pmatrix} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix}.$$

The determinant $\det \boldsymbol{X} = \frac{i\sqrt{b}\sqrt{a}}{b+a} + \frac{i\sqrt{a}\sqrt{b}}{b+a} = 2i\frac{\sqrt{ab}}{b+a}$. The inverse reduces to

$$\frac{1}{\sqrt{b+a}} \frac{a+b}{2i\sqrt{ab}} \begin{pmatrix} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix} = \frac{-i\sqrt{b+a}}{2\sqrt{ab}} \begin{pmatrix} i\sqrt{a} & -\sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix} = \frac{\sqrt{b+a}}{2\sqrt{ab}} \begin{pmatrix} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{pmatrix}.$$

We have now obtained the eigenvalue decomposition of $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$. To solve the linear ODE, we need to compute the product

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \boldsymbol{X} e^{\boldsymbol{D}t} \boldsymbol{X}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\
&= \frac{1}{\sqrt{b+a}} \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{pmatrix} \begin{pmatrix} e^{i\sqrt{ab}t} & 0 \\ 0 & e^{-i\sqrt{ab}t} \end{pmatrix} \frac{\sqrt{b+a}}{2\sqrt{ab}} \begin{pmatrix} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i\sqrt{a} & i\sqrt{a} \end{pmatrix} \begin{pmatrix} e^{i\sqrt{ab}t} & 0 \\ 0 & e^{-i\sqrt{ab}t} \end{pmatrix} \begin{pmatrix} \sqrt{a} & i\sqrt{b} \\ \sqrt{a} & -i\sqrt{b} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}e^{i\sqrt{ab}t} & \sqrt{b}e^{-i\sqrt{ab}t} \\ -i\sqrt{a}e^{i\sqrt{ab}t} & i\sqrt{a}e^{-i\sqrt{ab}t} \end{pmatrix} \begin{pmatrix} \sqrt{a}x_0 + i\sqrt{b}y_0 \\ \sqrt{a}x_0 - i\sqrt{b}y_0 \end{pmatrix}.
\end{aligned}$$

To simplify the last steps of the calculation, we will introduce the following notation. Using Euler's formula, we have $e^{\pm i\sqrt{ab}t} = \cos(\sqrt{ab}t) \pm i\sin(\sqrt{ab}t)$. Let the parameters c, s, C_1 and C_2 be $c = \cos(\sqrt{ab}t)$, $s = \sin(\sqrt{ab}t)$, and $C_1 = \sqrt{a}x_0 + i\sqrt{b}y_0$, $C_2 =$

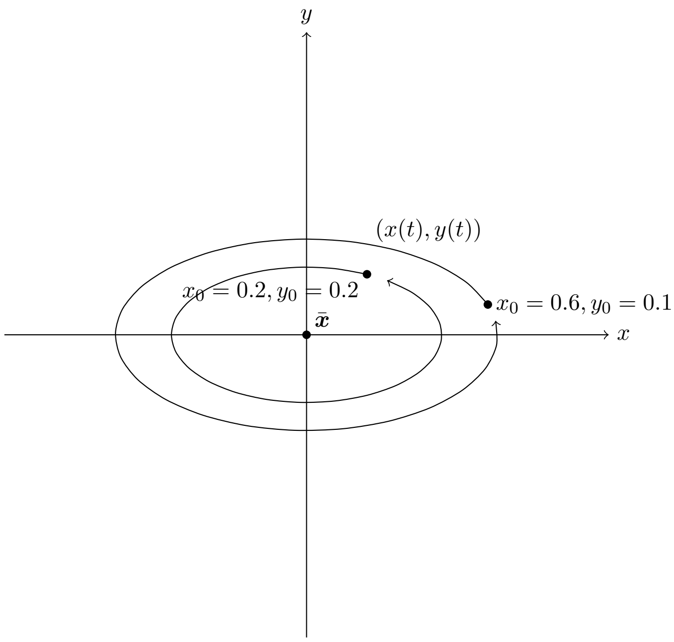


Figure 11. Solution of the linear system of ODEs (1), with $a = 0.1$, $b = 0.4$.

$\sqrt{a}x_0 - i\sqrt{b}y_0$. The solution now reads in a more compact manner:

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}e^{i\sqrt{ab}t}C_1 + \sqrt{b}e^{-i\sqrt{ab}t}C_2 \\ -i\sqrt{a}e^{i\sqrt{ab}t}C_1 + i\sqrt{a}e^{-i\sqrt{ab}t}C_2 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}(c+is)C_1 + \sqrt{b}(c-is)C_2 \\ -i\sqrt{a}(c+is)C_1 + i\sqrt{a}(c-is)C_2 \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} \sqrt{b}c(C_1+C_2) + i\sqrt{b}s(C_1-C_2) \\ \sqrt{a}s(C_1+C_2) + i\sqrt{a}c(-C_1+C_2) \end{pmatrix}, \\
&= \frac{1}{2\sqrt{ab}} \begin{pmatrix} 2\sqrt{ab}\cos(\sqrt{ab}t)x_0 - 2b\sin(\sqrt{ab}t)y_0 \\ 2a\sin(\sqrt{ab}t)x_0 + 2\sqrt{ab}\cos(\sqrt{ab}t)y_0 \end{pmatrix}, \\
&= \begin{pmatrix} \cos(\sqrt{ab}t)x_0 - \sqrt{b/a}\sin(\sqrt{ab}t)y_0 \\ \sqrt{a/b}\sin(\sqrt{ab}t)x_0 + \cos(\sqrt{ab}t)y_0 \end{pmatrix}.
\end{aligned}$$

And that's it! We have obtained a solution to the linear ODE (Figure 11).

11 Application to life sciences

11.1 Fibonacci sequence

The **Fibonacci sequence** is a sequence of numbers in which each element is the sum of the two preceding elements. Elements of the sequence are called **Fibonacci numbers**, usually denoted F_n. Starting with numbers 0 and 1, the first few numbers of the sequence are

Fibonacci sequence

Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

The Fibonacci sequence can be defined by a **recurrence relation**:

recurrence relation

$$F_0 = 0, F_1 = 1$$

and

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

The sequence can be used to calculate the growth of an idealised rabbit population:

- The population starts at month 1 with a newborn breeding pair of rabbit.
- Each breeding pair mates once a month starting at age of one month.
- Each breeding pair produces a breeding pair one month later.

Exercise 49 (Counting rabbits). How many pairs will there be a the end of month 12? *(Solution to exercise 49)*

Exercise 50 (Matrix form). Express the recurrence relation in a matrix form:

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = A \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix},$$

where A is a 2×2 matrix to be determined. *(Solution to exercise 50)*

Denote the vector $(F_{n+1}, F_n)^t$ by $\boldsymbol{F}_n$. The matrix form takes the simpler form $\boldsymbol{F}_{n+1} = A\boldsymbol{F}_n$, for $n \geq 0$, and $F_0 = (0, 1)^t$.

Exercise 51 (Recurrence formula). A solution for $\boldsymbol{F}_n$ can be expressed in terms of the initial conditions F_0 only. Use the recurrence relation to find an equation between $\boldsymbol{F}_n$ and F_0. *(Solution to exercise 51)*

The solution for $\boldsymbol{F}_n$ can therefore be obtained if we can compute the powers of the matrix A. If the matrix admit an eigenvalue decomposition $A = XDX^{-1}$, we can compute the n-th power $A = XD^nX^{-1}$.

Exercise 52 (Eigenvalues). Compute the eigenvalues of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Solution to exercise 52)

The positive eigenvalue called **golden ratio**, and is denoted ϕ. The other eigenvalue is denoted ψ. Nice properties of ϕ, ψ are: $\phi + \psi = 1$ and $\psi\phi = -1$. There are direct consequences of the form of the characteristic polynomial.

golden ratio

Exercise 53 (Eigenvalues). Compute the eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Solution to exercise 53)

A basis matrix X is

$$X = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}.$$

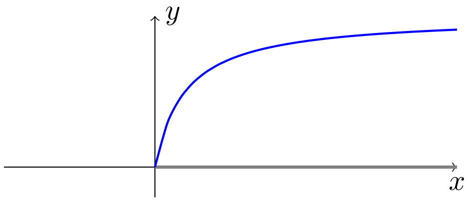
Exercise 54 (Eigenvalues). Compute the inverse of X *(Solution to exercise 54)*

Using $A^n = X\text{diag}(\phi^n, \psi^n)^t X^{-1}$, we can compute F_n explicitely.

Exercise 55 (Eigenvalues). Compute F_n. *(Solution to exercise 55)*

12 Solutions to the exercises

Solution to exercise 1 The graph of f, $f(x) = \frac{x}{K+x}$, is



Solution to exercise 2 The map $x \to -x^3$ is decreasing over $x \in \mathbb{R}$. The map $x \to \frac{x^2}{1+x^2}$ is increasing for $x \geq 0$. The other maps are not strictly increasing or decreasing.

Solution to exercise 3 $f(x) = x^3 - 3x$. The linear term $-3x$ dominates around zero. This term has a negative coefficient, so f is decreasing around $x = 0$. The higher order term x^3 dominates away from zero. As this term has a positive coefficient 1, the function f is increasing when x is far from zero. The function f is therefore neither increasing or decreasing on $\mathbb{R}$.

$f(x) = e^{-x^2}$. The argument of the exponential is $-x^2$, a quadratic term. It is increasing when $x \leq 0$ and decreasing when $x \geq 0$. The exponential function e^y is always increasing. Therefore the composed function $f(x) = e^{-x^2}$ increases when $x \leq 0$ and decreases when $x \geq 0$.

$f(x) = \ln(x)$. Increasing.

Solution to exercise 4 $f(x) = \sin(x) + \cos(x)$. The sine and cosines are bounded by -1 and 1. Their sum must be bounded by -2 and 2. However, they cannot be both equal to 1 or -1, so we can find better bounds. A symmetry argument suggest that a bound may be achieved when they are both equal, at $x = \pi/4$: $f(\pi/4) = \sqrt(2)/2 + \sqrt(2)/2 = \sqrt(2)$ Likewise, the lower bound would be $-\sqrt{2}$.

$f(x) = \frac{1}{x}$. (i) Close to 0, for any value $B > 0$, we can chose $x = B/2$, so that $f(x) = 2B$. Therefore f in not bounded. (ii) If $x > 1$, then $1/x < 1$. Moreover, $1/x > 0$. Therefore f is bounded below by 0 et above by 1.

$f(x) = -5x + 6$. This is a straight line with a negatve slope. Any real value y can be reach: $y = -5x + 6$ implies $x = (-y + 6)/5$. Thus there is no bound.

Solution to exercise 5

$$f(x) = \frac{x^2-1}{x-1}.$$

At $x = 1$, the denominator $x - 1$ is zero, indicating a potential problem. However, the numerator $x^2 - 1$ is also zero, so this is a undeterminate form, and we need to check how in fact the function behaves around 1. One way to do this is taking the limit of f as $x \to 1$:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

We can use the L'Hospital rule to compute the limit. The rule says that the limit remains the same if we take the derivatives of the numerators and the denominators:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{2x}{1} = 1.$$

The other way to go is too try to simplify the fraction. The numerator can be rewritten as $x^2 - 1 = (x - 1)(x + 1)$. This uses the identity rule for the difference of squares: $b^2 - a^2 = (b - a)(b + a)$. Once $x - 1$ appears on the numerator, it can be cancelled with the denominator, leaving $f(x) = x + 1$. Cancelling the terms $x - 1$ is allowed even when $x = 1$.

$f(x) = |x|$ is continuous everywhere, even at 0.

$$f(x) = \begin{cases} x^2, & x \le 2, \\ 3x-2, & x > 2 \end{cases}$$

At $x = 2$ we need to check that both sides have the same value: 2^2 and $3 \cdot 2 - 2$. The answer is yes, so f is continuous. This argument relies on the need for the limits when $x \to 2$ from the left and the right to be equal. Even if at $x = 2$, f is not defined as $3x - 2$, the limits should match.

Solution to exercise 6

$$\lim_{x\to\infty} \frac{2x+3}{x+5}.$$

Here we have a rational function, that is, a ratio of two polynomials. The behavior at infinity is dictated by the highest order terms: $2x$ at the numerator, and x at the denominator. The ratio is $2x/x = 2$, and this is the limit.

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}.$$

In order to have a horizontal asymptote, the function should have a limit at $+\infty$ or $-\infty$. Does it? No. To show that there is no limit, we can check that no matter how large x is, the function can go to infinity. If X is any real value (a large one), then we can choose an integer k, such that $x = 2k\pi + \pi/2 > X$. At that point, f(x) diverges to $+\infty$. This is enough to show that there is no limit, hence no asymptote.

$f(x) = \frac{\sin x}{x}$. As $x \to 0$, the numerator behaves like x, so the limit is just $x/x = 1$. As $x \to \infty$, the denominator goes to infinity, while the numerator remains bounded. The limit is of the form finite over infinite, and this is zero.

Solution to exercise 7 A function f is **odd** if it satisfies the property $f(x) = -f(-x)$. It is **even** if $f(x) = f(-x)$. odd even

$f(x) = x^4 + 1$. Polynomials with only even order terms are even. This one is even.

$f(x) = \frac{1}{1+x^2}$. This function is bounded: the denominator is at least one, so an upper bound is 1. It is also positive, so a lower bound is 0. It is continuous, and monotone decreasing over $[0, +\infty)$.

$f(x) = e^x$ is not bounded. For any value B, take $x = \ln(B) + 1$. At x, $f(x) = e^{\ln(B)+1} = Be^1 > B$. Therefore, no value is a bound. However, f is monotone: if $y > x$, then $f(y) = f([y-x]+x) = e^{[y-x]+x} = e^{[y-x]}e^x = e^{[y-x]}f(x)$, and because $y - x > 0$, $e^{[y-x]} > 1$, and $f(y) > f(x)$.

Solution to exercise 8 $f(x) = x^2 - 5x + 6$. This is a quadratic equation, and to solve it, we use the **quadractic formula**: the two roots x_1, x_2 of the quadratic polynomial $ax^2 + bx + c$ are quadractic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This assumes $a \neq 0$. For f, the roots are $(5 \pm \sqrt{25 - 4 \cdot 6})/2 = (5 \pm 1)/2$, that is, 2 and 3.

$f(x) = e^x + 2$ has no real zero: it is bounded below by 2.

$f(x) = \sin(x) - \frac{1}{2} = 0$ if $\sin(x) = \frac{1}{2}$. The sine function that infinitely many times any value between -1 and 1. It takes the value $\frac{1}{2}$ at $x = 30$ deg or $x = \frac{\pi}{6}$ rad, and at $x = 150$ deg or $x = \frac{5\pi}{6}$ rad. It will also take the value $\frac{1}{2}$ when these values are shifted by any number of 360 deg or 2π rad: $x = \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi$ for any integer k...

Solution to exercise 9 $f_1'(x) = -\frac{\sin x}{2\sqrt{\cos x}}$. $f_2'(x) = 3\cos(3x+2)$. $f_3'(x) = -\sin(x)e^{\cos x}$. $f_4'(x) = \frac{1}{2x}$. $f_5'(x) = \frac{\ln 2}{x} 2^{\ln x}$.

Solution to exercise 10 $f(x) = x^3 - 3x^2 + 4$. Extrema are found when $f' = 0$ and $f'' \neq 0$. The first derivative is $f'(x) = 3x^2 - 6x$, and it vanishes when $x = 0$ or when $x = 2$. The second derivative $f''(x) = 6x - 6$, and it is negative at $x = 0$ and positive at $x = 2$. Therefore the points $x = 0$ and $x = 2$ are extrema: $x = 0$ is a maximum, and $x = 2$ is a minimum.

$f(x) = \cos(x)$. This function reaches a maximum at $x = 0$ (or $x = 2\pi$) and a minimum at $x = \pi$.

$f(x) = \ln(x)$. The first derivative is $f'(x) = \frac{1}{x}$. On $(0, \infty)$, it never vanishes, and therefore does not have a minimum or a maximum.

Solution to exercise 11 Inflection points are points where the second derivative changes sign.

$f(x) = x^3 - 6x^2 + 9x$. The second derivative is $f''(x) = 6x - 12$, and it vanishes at $x = 2$. There is a change of sign, because the second derivative is linear.

$f(x) = e^x$ has no inflection point, because its second derivative, $f''(x) = e^x$ never vanishes.

$f(x) = \tan(x)$. The first derivative if $f'(x) = 1 + tan^2(x)$, and the second derivative $f''(x) = 2tan(x)[1 + tan^2(x)]$. The term in the square bracket is always positive, so the only way the second derivative can vanish is if and only if $tan(x) = 0$. It does when $x = k\pi$, and these are inflection points.

Solution to exercise 12 $f(x) = e^x$. All derivatives are e^x. At $x = 0$, this is 1, so the expansion is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

$f(x) = \sin(x)$. Even order derivatives evaluated at 0 are 0. Odd order derivatives evaluated at 0 are ± 1.

$$0 + x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

$f(x) = \ln(1 + x)$. We have $f(0) = \ln(1) = 0$, $f'(0) = 1$, $f''(0) = -1$, and $f'''(0) = \frac{2}{(1+0)^3} = 2$. The expansion to order 3 is

$$0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$

Solution to exercise 13 $f(x) = \cos(x)$ around $x = \pi/2$. We have $f(\pi/2) = 0$, $f'(\pi/2) = -1$, and $f''(\pi/2) = 0$. The expansion up to order 2 is

$$0 - (x - \pi/2) + 0 = x - \pi/2.$$

$f(x) = \sqrt{1 + x}$ around $x = 1$. We have $f(1) = \sqrt{2}$, $f'(1) = \frac{1}{2\sqrt{2}}$, $f''(1) = -\frac{1}{42^{3/2}}$, and $f'''(1) = \frac{3}{82^{5/2}}$. The expansion is

$$\sqrt{2} + \tfrac{1}{2\sqrt{2}}(x - 1) - \tfrac{1}{82^{3/2}}(x - 2)^2 + \tfrac{3}{482^{5/2}}(x - 1)^3.$$

$f(x) = \ln(1 + x)$ around $x = 2$. We have $f(2) = \ln(3)$, $f'(2) = \frac{1}{1+2} = \frac{1}{3}$, and $f''(2) = \frac{-1}{(1+2)^2} = \frac{-1}{9}$. The expansion to order 2 is

$$\ln(3) + \tfrac{1}{3}(x - 2) - \tfrac{1}{18}(x - 2)^2.$$

Solution to exercise 14 Approximate $\ln(1 + x)$. The n-th derivative of $\ln(1 + x)$ is $\frac{(-1)^(n-1)(n-1)!}{1+x^n}$. Evaluated at zero, we have $(-1)^(n - 1)(n - 1)!$. The expansion is

$$0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...$$

We want $\ln(1.1) = f(0.1)$. How large is the error if we expand up to order n? Because of the alternate signs, the error is less than the absolute value of the first term dropped. If we expand up to order 2, the error should be less than the third order term: $\frac{1}{3}0.1^3 = 0.3 \times 10^{-3}$. The 2nd order approximation is $0.1 - \frac{1}{2}0.1^2 = 0.1 - 0.005 = 0.095$. The calculator gives a value of 0.09531... for an error of 0.0003.

sin(0.1). A third order expansion is

$$0 + x - \frac{x^3}{6}.$$

At $x = 0.1$, we have a value of $0.1 - \frac{0.001}{6} = 0.099833$. The error is bounded by the next non-zero term: $\frac{x^5}{5!}$, which evaluates to 8.333333×10^{-8}, so a conservative error would be 10^{-7}.

$f(x) = e^x$. The 2nd order expansion is $1 + x + x^2/2$. At $x = 2$, this evaluates to $1 + 0.5 + 0.125 = 1.625$. The true value is 1.648721.

Solution to exercise 15 The remainder of the expansion of e^x at order n is

$$\sum_{k=n+1}^{\infty} \frac{x^k}{k!}.$$

With $n = 5$ and $x = 1$, the error is

$$\sum_{k=6}^{\infty} \frac{1}{k!}.$$

Here we want to bound the error from above, not to compute it exactly. Clearly, since $k \geq 6$, we can replace $k!$ by $k \cdot (k-1) \cdot ... 7 \cdot 6!$, which is greater than $7 \cdot 7 \cdot ... 7 \cdot 6! = 7^{k-6} 6!$. Therefore

$$\sum_{k=6}^{\infty} \frac{1}{k!} < \frac{1}{6!} \sum_{k=6}^{\infty} \frac{1}{7^{k-6}}.$$

The sum a geometrical series with a reason of $r = \frac{1}{7}$, and the sum is $\frac{1}{1-r} = \frac{7}{6}$. We have found a bound: $\frac{7}{66!} = 0.00162$.

The error of evaluating ln(1.5) using a third order expansion of ln$(1 + x)$ is less than the fourth degree term in absolute value in the expansion (see previous exercise): $\frac{1}{4}0.5^4 = 0.015625$. The approximation is $0.5 - \frac{1}{2}0.5^2 + \frac{1}{3}0.5^3 = 0.416667$. The true value is 0.405465... The difference is 0.011202, within the bound.

Solution to exercise 16

$$f(x, y) = x^2 + y^2.$$

The function is already in the form of a Taylor expansion of order 2 around zero. We can still compute it to check:

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)xy + ...$$

$$= 0 + 2 \cdot 0 \cdot x + 2 \cdot 0y + x^2 + x^2 + 0 \cdot xy$$

Here the expansion is exact, because higher order terms are all zero.

$f(x,y)=e^{x+y}$. The expansion around zero is

$$e^0+e^0x+e^0y+\frac{1}{2}e^0x^2+\frac{1}{2}e^0y^2+e^0xy.$$

$$=1+x+y+\frac{1}{2}x^2+\frac{1}{2}y^2+xy$$

$$f(x,y)=\sin(x)\cos(y).$$

The expansion is

$$0+1\cdot x+0\cdot y+\frac{1}{2}0\cdot x^2+\frac{1}{2}0y^2+0\cdot xy+\frac{1}{3!}(-1)x^3+\frac{1}{3!}0y^3+\frac{3}{3!}0x^2y+\frac{3}{3!}(-1)xy^2$$

$$=x-\frac{x^3}{6}-\frac{xy^2}{2}.$$

For instance, $f(0.1,0.2)\approx 0.1-\frac{0.1^3}{6}-\frac{0.10.2^2}{2}=0.097833$. The true value is $0.097843\ldots$.

Solution to exercise 17 $f(x,y)=\ln(1+x+y)$. The expansion is

$$\ln(1+0+0)+\frac{1}{1+0+0}x+\frac{1}{1+0+0}y+\frac{1}{2}\frac{-1}{(1+0+0)^2}x^2+\frac{1}{2}\frac{-1}{(1+0+0)^2}y^2+\frac{-1}{(1+0+0)^2}xy,$$

$$=x+y-\frac{1}{2}x^2+-\frac{1}{2}x^2-xy.$$

$f(x,y)=\sqrt{1+x^2+y^2}$. The expansion is

$$\sqrt{1}+\frac{2\cdot 0}{2\sqrt{1}}x+\frac{2\cdot 0}{2\sqrt{1}}y+\frac{1}{2}\frac{1}{2}x^2+\frac{1}{2}\frac{1}{2}x^2+-\frac{0\cdot 0}{(1)^{3/2}}xy.$$

$$1+\frac{1}{4}x^2+\frac{1}{4}y^2.$$

At $0.1,0.2$, the expansion is $1+\frac{1}{4}0.1^2+\frac{1}{4}0.2^2=1.0125$, while the true value is $1.024695\ldots$.

$f(x,y)=e^x\sin(y)$ at $(1,0)$. The expansion is

$$0+0\cdot(x-1)+e^1\cdot y+\frac{1}{2}0\cdot(x-1)^2\frac{1}{2}0\cdot y^2+e^1\cdot(x-1)y$$

$$ey+e(x-1)y.$$

At $(1.1,0.2)$, the approximation is $e0.2+e(1.1-1)0.2=0.598022$, while the true value is 0.596836.

Solution to exercise 18 TODO

Solution to exercise 19 TODO

Solution to exercise 20

- $\int \frac{1}{\sqrt{3x+5}}dx$. By the change of variable $u = 3x+5, du = 3dx$, we have

$$\begin{aligned}\int \frac{1}{\sqrt{3x+5}}dx &= \int \frac{1}{\sqrt{u}}\frac{du}{3},\\ &= \frac{1}{3}\frac{u^{\frac{1}{2}}}{\frac{1}{2}},\\ &= \frac{2}{3}u^{\frac{1}{2}},\\ &= \frac{2}{3}(3x+5)^{\frac{1}{2}}.\end{aligned}$$

- $f(x) = \cos(2x)$. Use a simple substitution or recall that derivative of $\sin(2x)$ is $2\cos(2x)$. Thus

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.$$

- $\int (2x)\,e^{x^2}\,dx$. Let $u = x^2$. Then $du = 2x\,dx$. Hence

$$\int 2xe^{x^2}\,dx = \int e^u\,du = e^u + C = e^{x^2} + C.$$

- $\int \frac{1}{x\ln x}\,dx$. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$. So

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C.$$

- Compute $\int x\sin x\,dx$. Use integration by parts. Let

$$u = x, \quad dv = \sin x\,dx \quad \Rightarrow \quad du = dx, \quad v = -\cos x.$$

Then

$$\int x\sin x\,dx = uv - \int v\,du = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

- $\int \frac{1}{x^2-1}\,dx$ (write the result using partial fractions). Factor denominator: $x^2 - 1 = (x-1)(x+1)$. Partial fractions:

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Solve:

$$1 = A(x+1) + B(x-1).$$

Set $x = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$. Set $x = -1$: $1 = -2B \Rightarrow B = -\frac{1}{2}$. So

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

This can be combined as $\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$.

- $\displaystyle\int \frac{2x+3}{x^2+3x+2}\,dx$. Factor denominator: $x^2+3x+2 = (x+1)(x+2)$. Try partial fractions or notice derivative of denominator:

$$\frac{d}{dx}(x^2+3x+2) = 2x+3,$$

which matches the numerator exactly. Thus

$$\int \frac{2x+3}{x^2+3x+2}\,dx = \int \frac{(x^2+3x+2)'}{x^2+3x+2}\,dx = \ln|x^2+3x+2| + C.$$

- $\displaystyle\int \sin^2 x\,dx$. Use the identity $\sin^2 x = \frac{1}{2}(1-\cos(2x))$:

$$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1-\cos(2x))\,dx = \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin(2x)\right) + C.$$

Optionally rewrite $\sin(2x) = 2\sin x\cos x$ to get

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin x\cos x}{2} + C.$$

- $\displaystyle\int \frac{dx}{(x+1)^2}$. This is a standard power rule. Let $u = x+1$. Then $du = dx$ and integral becomes

$$\int u^{-2}\,du = -u^{-1} + C = -\frac{1}{x+1} + C.$$

- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$. Recognize derivative of $\arcsin x$. For $|x| < 1$,

$$\frac{d}{dx}\big(\arcsin x\big) = \frac{1}{\sqrt{1-x^2}}.$$

Thus

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

- $\displaystyle\int \frac{x^2}{(x+1)^2}\,dx$. We can simplify by division or rewrite integrand. Perform polynomial division or write

$$\frac{x^2}{(x+1)^2} = \frac{(x+1-1)^2}{(x+1)^2} = \left(1 - \frac{1}{x+1}\right)^2.$$

Expand:

$$\left(1 - \frac{1}{x+1}\right)^2 = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}.$$

Integrate termwise:

$$\int \frac{x^2}{(x+1)^2}\,dx = \int \left(1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}\right) dx.$$

Compute each:

$$\int 1\,dx = x, \qquad \int -\frac{2}{x+1}\,dx = -2\ln|x+1|, \qquad \int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1}.$$

Combine:

$$\int \frac{x^2}{(x+1)^2}\,dx = x - 2\ln|x+1| - \frac{1}{x+1} + C.$$

(You may check by differentiating.)

Solution to exercise 21 The integrand is the product of two functions, so let us try integration by parts. Let $g(x) = x$ and $f'(x) = e^x$. Then $g'(x) = 1$ and $f(x) = e^x$. The integral becomes

$$\begin{aligned}\int_0^1 xe^x dx &= e^x x\Big|_0^1 - \int_0^1 e^x dx,\\ &= e^1 - 0 - e^x\Big|_0^1,\\ &= e - (e^1 - e^0),\\ &= 0 + 1,\\ &= 1.\end{aligned}$$

Solution to exercise 22 The integrand is the product of two functions, we try integration by parts. Let $g(x) = \cos x$ and $f'(x) = e^x$. Then $g'(x) = -\sin x$ and $f(x) = e^x$. The integral becomes

$$\begin{aligned}\int_0^\pi \cos x e^x dx &= e^x(\cos x)\Big|_0^\pi - \int_0^\pi e^x(-\sin x)dx,\\ &= -e^\pi - e^0 - \int_0^\pi e^x(-\sin x)dx,\\ &= -e^\pi - 1 - \int_0^\pi e^x(-\sin x)dx,\end{aligned}$$

We still have a integral of a product to compute. We apply the integral by part once more,

$$\begin{aligned}&= -e^\pi - 1 - \left(e^x(-\sin x)\Big|_0^\pi + \int_0^\pi e^x(\cos x)dx\right).\\ &= -e^\pi - 1 - \int_0^\pi e^x(\cos x)dx.\end{aligned}$$

With this new integration, we just came back to our initial integral. We are looping, and further integration by part will not help us. To break the loop, we use the fact that the initial integral term is present on both side of the equation and solve for it. If

$$I = \int_0^{\pi} \cos x e^x dx,$$

then

$$\begin{aligned} I &= -e^{\pi} - 1 - I, \\ 2I &= -e^{\pi} - 1, \\ I &= \frac{-e^{\pi} - 1}{2}. \end{aligned}$$

Et voilà!

Solution to exercise 23 With the identity $\sinh x = \frac{e^x - e^{-x}}{2}$,

The integrand simplifies to $\frac{e^{2x}-1}{2}$, and

$$\int_0^1 \frac{e^{2x}-1}{2} dx = \frac{1}{2}[\tfrac{e^{2x}}{2} - x]|_0^1 = [\frac{1}{4}e^2 - \frac{1}{2}] - [\frac{1}{4}] = \frac{1}{4}e^2 - \frac{3}{4}.$$

Solution to exercise 24

1. linear, non-autonomous, not in normal form
2. non-linear, non-autonomous, not in normal form
3. non-linear, autonomous, not in normal form
4. non-linear, autonomous, not in normal form
5. non-linear, non-autonomous, not in normal form

Solution to exercise 25 The initial condition $x(0) = 1$ tells us to start the integration at $t = 0$. Using the second method with $a(t) = 2$ and $b(t) = 6$, we have

$$x(t) = Ke^{-\int_0^t -\frac{6}{2} du} = Ke^{-\frac{6}{2}t}.$$

The constant K is determined by the initial condition,

$$x(0) = Ke^0 = K = 1.$$

The complete solution is

$$x(t) = e^{-\frac{6}{2}t}.$$

Solution to exercise 26 The initial condition $x(1) = 1$ tells us to start the integration at $t = 1$. Using the second method with $a(t) = 1$ and $b(t) = 1/t$, we have

$$x(t) = Ke^{-\int_1^t -\frac{1}{t} du} = Ke^{-\ln u|_1^t} = Ke^{-\ln t}.$$

The term $e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$. The solution simplifies to

$$x(t) = \frac{K}{t}.$$

The constant K is found with the initial condition $x(1) = K = 1$, for a complete solution $x(t) = \frac{1}{t}$.

Solution to exercise 27 $\bar{z} = 2 - 3i$, $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$, $|\bar{z}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$, we see that $|z| = |\bar{z}|$, $\Re(\bar{z}) = 2$, $\Im(\bar{z}) = -3$, $\frac{z+\bar{z}}{2} = (2 + 3i + (2 - 3i))/2 = 2$, $\frac{z-\bar{z}}{2} = ((2 + 3i - (2 - 3i))/2 = 3i$, $-z = -2 - 3i$, $iz = 2i + 3i^2 = -3 + 2i$.

Solution to exercise 28 The modulus of z is $|z| = \sqrt{r^2 \cos^2(\theta) + r^2 \sin^2(\theta)} = \sqrt{r^2} = r$. From Euler's formula, we have $\cos(\theta) + i\sin(\theta) = e^{i\theta}$, so $z = re^{i\theta}$. Therefore, for any complex number $z = re^{i\theta}$, $|z| = 1$ if and only if $r = 1$.

Solution to exercise 29 All trigonometric identities can be obtained by applying Euler's formula. Here we start from $e^{ia+ib} = \cos(a + b) + i\sin(a + b)$. We only want the real part,

$$\begin{aligned}
\cos(a+b) &= \frac{e^{ia+ib} + e^{-ia-ib}}{2} \\
&= \frac{e^{ia}e^{ib} + e^{-ia}e^{-ib}}{2} \\
&= \frac{(\cos(a) + i\sin(a))(\cos(b) + i\sin(b))}{2} \\
&= + \frac{(\cos(a) - i\sin(a))(\cos(b) - i\sin(b))}{2} \\
&= \frac{\cos(a)\cos(b) + i^2\sin(a)\sin(b) + i\cos(a)\sin(b) + i\cos(b)\sin(a)}{2} \\
&\quad + \frac{\cos(a)\cos(b) + i^2\sin(a)\sin(b) - i\cos(a)\sin(b) - i\cos(b)\sin(a)}{2}.
\end{aligned}$$

The mixed cosine-sine terms cancel each other while the other ones add up, resulting in

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Solution to exercise 30 This is a direct application of Euler's formula: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

Solution to exercise 31 The roots must satisfy $e^{i6\theta} = 1$. This means that $\theta = \frac{2}{6}k\pi$, for $k = 0, 1, ..., 5$. There are six distinct roots.

Solution to exercise 32 The exponential converges to zero if and only if $\Re(z) < 0$. A complex number is close to zero if and only if its modulus is close to zero. Therefore, to show that a quantity converges to zero, it is necessary and sufficient to show that its modulus converges to zero. If $z = a + ib$, the exponential $e^{zt} = e^{(a+ib)t} = e^{at}e^{ibt}$.

The modulus $|e^{ibt}| = 1$, so $|e^{zt}| = e^{at}$ (no need for absolute values, the exponential of a real number is always positive). The condition for convergence to zero is therefore a condition on the real part of z: $e^{at} \to 0$ when $t \to \infty$ if and only if $a < 0$.

Solution to exercise 33 The square root of a complex number always exists. Express z in polar form $z = re^{i\theta}$, $r \geq 0, \theta \in [0, 2\pi]$. The square root $\sqrt{z} = \sqrt{r}\sqrt{e^{i\theta}} = \sqrt{r}e^{\frac{1}{2}i\theta}$. Using Euler's formula, $\sqrt{z} = \sqrt{r}\cos(\theta/2) + i\sqrt{r}\sqrt{\theta/2}$. That is, the square root is obtained by taking the square root of the modulus r, and dividing the angle (the **argument**) by 2. There is a problem with this solution, because z can also be represented by $re^{i\theta+2\pi}$, giving $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta+\pi}$, which is equivalent to dividing the angle by two in the other direction. We define the **principal square root** as the solution that makes the smallest change in angle: $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta}$ if $\theta \in [0, \pi]$, and $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta+\pi}$ if $\theta \in (\pi, 2\pi]$ To express the solution in terms of the original form of $z = a + ib$, we express the square root $s = \alpha + i\beta$. Then $s^2 = \alpha^2 - \beta^2 + 2i\alpha\beta = z = a+ib$. By identifying the real and imaginary parts, we get two equations: $\alpha^2 - \beta^2 = a$ and $2i\alpha\beta = b$. Denoting the modulus of z by $r = \sqrt{a^2 + b^2}$, we can obtain the solutions

argument

principal square root

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{a+r}, \quad \beta = \text{sign}(b)\frac{1}{\sqrt{2}}\sqrt{-a+r}.$$

Solution to exercise 34 TODO

Solution to exercise 35

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

The transformation is a 90 degree counterclockwise rotation.

Solution to exercise 36

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix exchanges the coordinates of a vector, this is a reflection through the axis $x = y$.

Solution to exercise 37

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The product is not the same, the matrices do not commute. The transformation is now a reflection through $y = -x$.

Solution to exercise 38 The matrix is

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Solution to exercise 39 The matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution to exercise 40 The matrix is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solution to exercise 41 The power of a diagonal matrix is a diagonal matrix

$$A^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}.$$

Solution to exercise 42 The scalar product is

$$\begin{aligned} x^* y &= (1-2i, 1+i)(0.5-i, -0.5)^t \\ &= (1-2i)(0.5-i) + (1+i)(-0.5) \\ &= 0.5 + 2i^2 - 2(0.5)i - i - 0.5 - 0.5i \\ &= (0.5 - 0.5 + 2i^2) + (-2(0.5) - 1 - 0.5)i \\ &= -2 - 2.5i. \end{aligned}$$

Solution to exercise 43 The scalar product is

$$\begin{aligned} y^* x &= (0.5+i, -0.5)(1+2i, 1-i)^t \\ &= (0.5+i)(1+2i) + (-0.5)(1-i) \\ &= 0.5 + 2i^2 + i + 2(0.5)i - 0.5 + 0.5i \\ &= -2 + 2.5i \end{aligned}$$

This is the conjugate: $x^* y = (y^* x)^*$.

Solution to exercise 44 The scalar product $z^* z = (\bar{z}_1, \bar{z}_2)(z_1, z_2)^t = \bar{z}_1 z_1 + \bar{z}_2 z_2 = |z_1|^2 + |z_2|^2$. The scalar product is the square of the norm of the vector z.

Solution to exercise 45 We have $\det A = (-1)(-1) - (2)(2) = 1 - 4 = -3 < 0$, $\operatorname{tr} A = -1 - 1 = -2$, and $\Delta = (-2)^2 - 4(-3) = 4 + 12 = 16$. The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(-2 \pm \sqrt{16}\right) = -1 \pm 2 = 1, -3.$$

The two eigenvalues are distinct, so the matrix A is diagonalisable. The eigenvector associated with the eigenvalue $\lambda_1 = 1$ is solution to the eigenproblem $\left(A - \lambda_1 I\right)x = 0$. We look for a solution

$$\begin{aligned} \begin{pmatrix} -1-(1) & 2 \\ 2 & -1-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= 0, \\ \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= 0. \end{aligned}$$

(We check that the two rows are co-linear.) The first row gives $-2x_1 + 2x_2 = 0$, or $x_1 = x_2$. The norm of the eigenvector $x = (x_1, x_2)^t = \sqrt{x_1^2 + x_2^2} = \sqrt{2}|x_1|$. We choose$x_1 = 1/\sqrt{2}$ to have a normalized eigenvector. We know that the second eigenvector is orthogonal to x, so we can take $y = (1/\sqrt{2}, -1/\sqrt{2})^t$ for the eigenvector associated to $\lambda_2 = -3$. To check that this is indeed an eigenvector, we solve to eigenproblem

$$\begin{pmatrix} -1-(-3) & 2 \\ 2 & -1-(-3) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0.$$

The solutions are vectors that satisfy $y_1 = -y_2$; this is the case for y. The matrix X is composed of the column vectors x and y: $X = \left(x|y\right)$ and its inverse is

$$X^{-1} = X^t = \begin{pmatrix} x_1^t \\ x_2^t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution to exercise 46 TODO

Solution to exercise 47 The steady states are found by solving

$$f_1(x,y) = -dx + x\exp(-axy) = 0,$$
$$f_2(x,y) = x - y = 0.$$

From the second equation, we have $x = y$. The first equation is equivalent to $dx = x\exp(-axy)$. We need to distinguish two cases: (i) $x = 0$, and (ii) $x \neq 0$. Case (i) leads to the solution $\boldsymbol{x}^* = (0,0)^t$, our first steady state. Case (ii) means that we can simplify x in the first equation: $d = \exp(-axy)$. Replacing $y = x$, and solving for x:

$$\begin{aligned} d &= \exp(-axy), \\ d &= \exp(-ax^2), \\ \ln d &= -ax^2, \\ -\frac{\ln d}{a} &= x^2, \quad (a > 0) \end{aligned}$$

The hypothesis $d < 1$ ensures that $\ln d < 0$ and $-\ln d > 0$. There are therefore two real solutions for x:

$$x = \pm\sqrt{-\frac{\ln d}{a}}.$$

The two additional steady states are

$$\bar{\boldsymbol{x}}_{1,2} = \begin{pmatrix} \pm\sqrt{-\frac{\ln d}{a}} \\ \pm\sqrt{-\frac{\ln d}{a}} \end{pmatrix}.$$

The Jacobian matrix of $\boldsymbol{f}$ is computed from the partial derivatives

$$\begin{aligned}
\frac{\partial f_1}{\partial x}(x,y) &= -d + (-ay)\exp(-axy),\\
\frac{\partial f_1}{\partial y}(x,y) &= (-ax)\exp(-axy),\\
\frac{\partial f_2}{\partial x}(x,y) &= 1,\\
\frac{\partial f_2}{\partial y}(x,y) &= -1.
\end{aligned}$$

$$\boldsymbol{J} = \begin{pmatrix} -d - ay\exp(-axy) & -ax\exp(-axy) \\ 1 & -1 \end{pmatrix}.$$

The function f_2 is linear. This is reflected in the Jacobian matrix, which as constant coefficients on the second row. The evaluation of the Jacobian matrix at steady state $\boldsymbol{x}^* = (0,0)^t$ is

$$\boldsymbol{J}(\boldsymbol{x}^*) = \begin{pmatrix} -d & 0 \\ 1 & -1 \end{pmatrix}.$$

The evaluation of Jacobian matrix at steady state $\bar{\boldsymbol{x}}_1 = \left(\sqrt{-\frac{\ln d}{a}}, \sqrt{-\frac{\ln d}{a}}\right)^t$ is

$$\boldsymbol{J}(\bar{\boldsymbol{x}}_1) = \begin{pmatrix} -d - a\bar{y}_1\exp(-a\bar{x}_1\bar{y}_1) & -a\bar{x}_1\exp(-a\bar{x}_1\bar{y}_1) \\ 1 & -1 \end{pmatrix}.$$

Here, we use the fact that steady states satisfy the equation $\exp(-axy) = d$ to simplify the exponential terms

$$\boldsymbol{J}(\bar{\boldsymbol{x}}_1) = \begin{pmatrix} -d - a\bar{y}_1 d & -a\bar{x}_1 d \\ 1 & -1 \end{pmatrix}.$$

Replacing y_1 and x_1 by $\sqrt{-\frac{\ln d}{a}}$, we obtain

$$\begin{aligned}
\boldsymbol{J}(\bar{\boldsymbol{x}}_1) &= \begin{pmatrix} -d - ay_1 d & -a\bar{x}_1 d \\ 1 & -1 \end{pmatrix},\\
&= \begin{pmatrix} -d - a\sqrt{-\frac{\ln d}{a}}d & -a\sqrt{-\frac{\ln d}{a}}d \\ 1 & -1 \end{pmatrix},\\
&= \begin{pmatrix} -d\left(1 + a\sqrt{-\frac{\ln d}{a}}\right) & -d\sqrt{-a^2\frac{\ln d}{a}} \\ 1 & -1 \end{pmatrix},\\
&= \begin{pmatrix} -d\left(1 + \sqrt{-a\ln d}\right) & -d\sqrt{-a\ln d} \\ 1 & -1 \end{pmatrix}.
\end{aligned}$$

The same lines of calculations for the steady state $\bar{\boldsymbol{x}}_2$ lead to

$$\boldsymbol{J}(\bar{\boldsymbol{x}}_2) = \begin{pmatrix} -d\left(1 - \sqrt{-a\ln d}\right) & d\sqrt{-a\ln d} \\ 1 & -1 \end{pmatrix}.$$

Solution to exercise 48 TODO

Solution to exercise 49 144 pairs.

Solution to exercise 50 F_{n+2} is the sum of the coefficients of the right-hand side vector, so the first row of A is $(1,1)$. F_{n+1} is just the first coefficient of the right-hand side vector, so the second row of A is $(1,0)$. The matrix A is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution to exercise 51 $\boldsymbol{F}_1 = A\boldsymbol{F}_0$, $\boldsymbol{F}_2 = A\boldsymbol{F}_1 = AA\boldsymbol{F}_0$, ... $\boldsymbol{F}_n = AA...A\boldsymbol{F}_0 = A^n\boldsymbol{F}_0$.

Solution to exercise 52 The characteristic polynomial of A is

$$\lambda^2 - \lambda - 1 = 0.$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Solution to exercise 53 The eigenvalues are

$$\phi = \frac{1+\sqrt{5}}{2}$$

and

$$\psi = \frac{1-\sqrt{5}}{2}.$$

The first eigenvector is $v_1 = (\phi, 1)^t$. The second eigenvector is $v_2 = (1, -\phi)^t$.

Solution to exercise 54 The inverse of X is

$$X^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\psi \\ -\psi & -1 \end{pmatrix}.$$

Solution to exercise 55

$$\begin{aligned}
\boldsymbol{F}_n &= X\mathrm{diag}(\phi^n, \psi^n)^t X^{-1}(1,0)^t, \\
\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \psi^n \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\psi \\ -\psi & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \psi^n \end{pmatrix} \begin{pmatrix} 1 \\ -\psi \end{pmatrix}, \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \psi^n \\ -\psi^{n+1} \end{pmatrix}, \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi^{n+1} - \psi^{n+1} \\ \phi^n - \psi^n \end{pmatrix}.
\end{aligned}$$

As a result, we have

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

13 Glossary

French	English	Note
dérivable	differentiable	
matrice jacobienne	Jacobian matrix	
ensemble	set	
espace vectoriel	vector space	
sous-espace vectoriel	linear subspace	
valeur propre	eigenvalue	
vecteur propre	eigenvector	
sous-espace propre	eigenspace	
décomposition en valeurs propres	eigenvalue decomposition	
trace	trace	tr
déterminant	determinant	det
application linéaire	linear map	
application	map	
dimension	dimension	
produit scalaire	scalar product	

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