Geomechanics– Materials - Structures

GEOMAS

UNIVERSITÉ DE LYON

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

Constitutive modelling of soils

Prof. Ali DAOUADJI

Department of Civil Engineering and Urban Planning

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Context

Foundation / bearing capacity & settlement

Slope stability / landslide

Wind turbine

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Outline

1. Basic concepts of continuum mechanics
2. Elasticity
    1. Linear elasticity
    2. Nonlinear elasticity
3. Perfect plasticity
4. Plasticity in Soil Mechanics
    1. Non linear plasticity
    2. Isotropic hardening
    3. Single yield surface plasticity

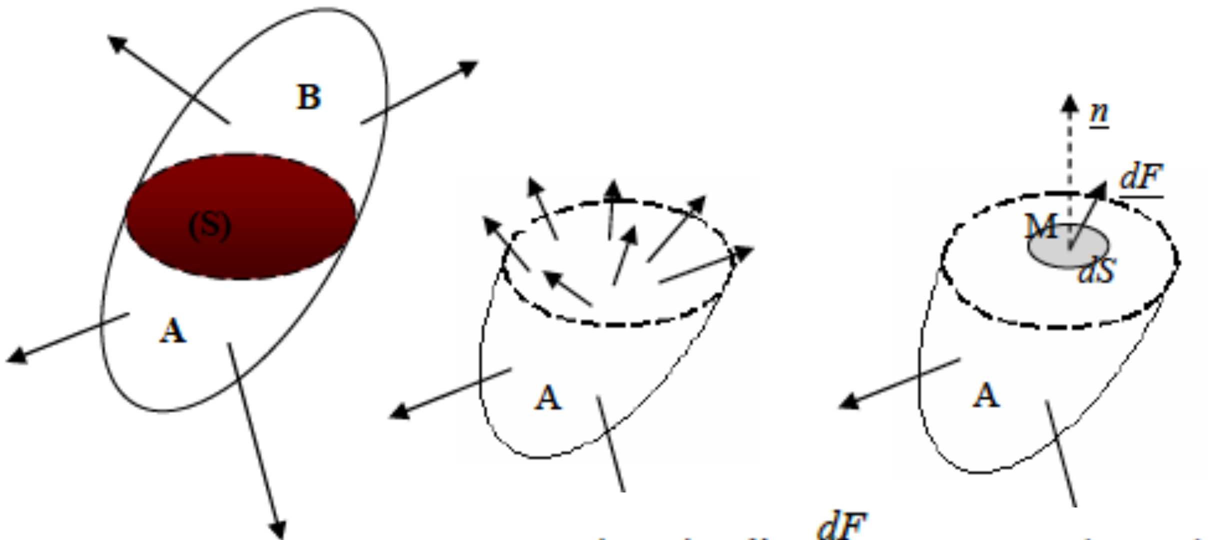
BASIC CONCEPTS OF CONTINUUM MECHANICS

1. Stress tensor
2. Equilibrium
3. Principal stresses and invariants
4. Strain tensor

1. Stress tensor

Stress state around a point:

Given a domain (D) occupied by a solid body



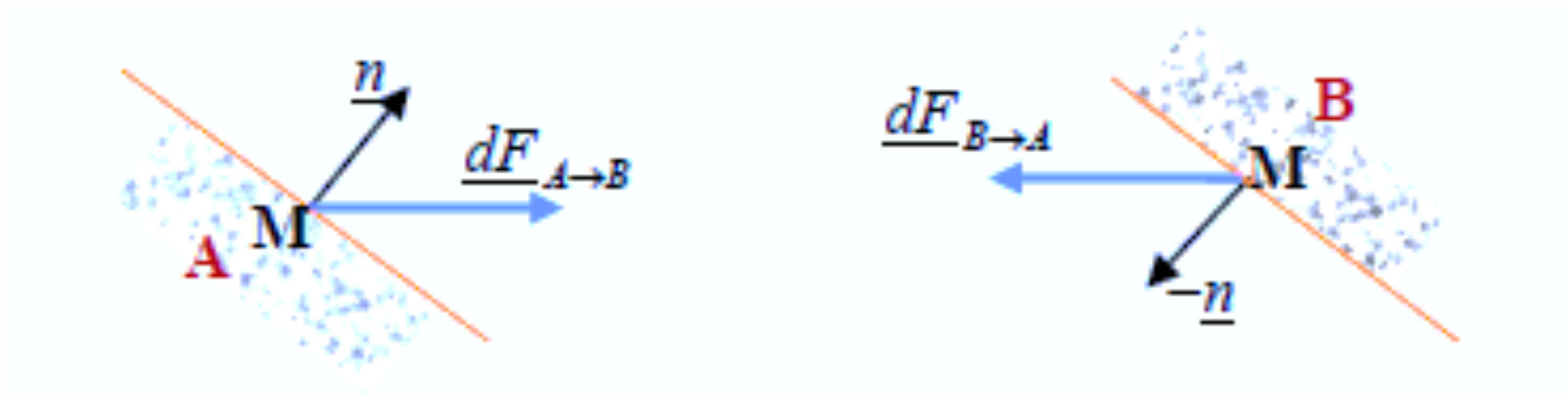
$$\underline{T}(M,\underline{n}) = \lim_{dS \to 0} \frac{\underline{dF}}{dS} \qquad \underline{T}(M,-\underline{n}) = -\underline{T}(M,\underline{n})$$

$\underline{dF}$ is the resultant of the elementary loads applying on the surface (S)

$\underline{T}(M,\underline{n})$ is defined as the stress vector and is given by

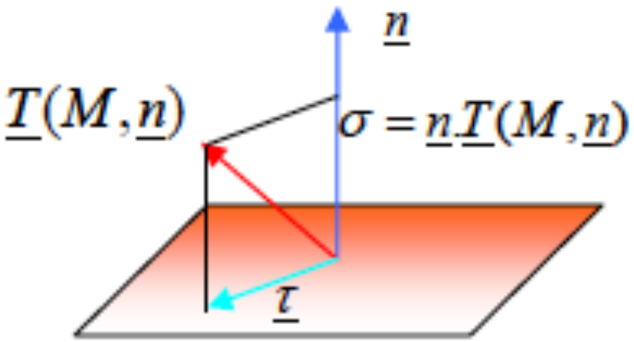
$$\underline{dF} = \underline{T}(M,\underline{n})dS$$

1. Stress Tensor



The magnitude of force acting on the part A at point M is equal and is acting in the opposite direction at point M belonging to the part B.

Basic concepts of CM



The stress vector T*(M,n)* can be projected on the axis oriented toward the external normal *n* and on the tangential axis *t.*

$$\underline{T}(M,\underline{n}) = \sigma \underline{n} + \underline{\tau}$$

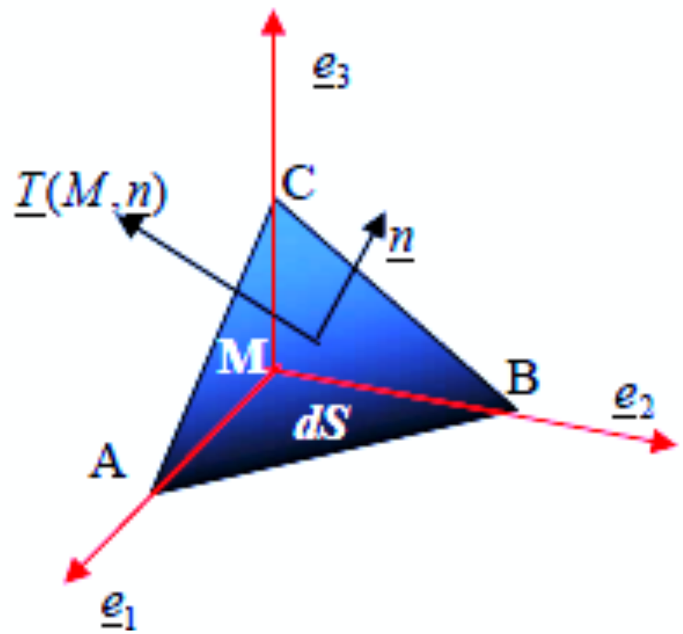
σ is the normal stress

$\underline{\tau}$ is the shear stress vector

$$\left\|\underline{T}(M,\underline{n})\right\|^2 = \sigma^2 + \left\|\underline{\tau}\right\|^2$$

1. Stress tensor

Around point M



orthonormal basis $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$

On: (ABC) normal $\underline{n}(n_1, n_2, n_3)$ $\underline{dF} = dS\,\underline{T}(M,\underline{n})$

(MBC) normal $-\underline{e}_1$, $\underline{dF_1} = dS_1 \underline{T_1}(M, -\underline{e}_1)$

(MAC) normal $-\underline{e}_2$, $\underline{dF_2} = dS_2 \underline{T_2}(M, -\underline{e}_2)$

(MAB) normal $-\underline{e}_3$, $\underline{dF_3} = dS_3 \underline{T_3}(M, -\underline{e}_3)$

1. Stress tensor

$$2dS\underline{n} = \underline{AB} \wedge \underline{AC} = (\underline{MB} - \underline{MA}) \wedge (\underline{MC} - \underline{MA}) = \underline{MB} \wedge \underline{MC} + \underline{MA} \wedge \underline{MB}$$

$$+ \underline{MA} \wedge \underline{MB} + \underline{MC} \wedge \underline{MA} = 2dS_1\underline{e}_1 + 2dS_2\underline{e}_2 + 2dS_3\underline{e}_3$$

Multiplying by e_1: $dS_1 = dSn_1$

Same for e_2 and e_3: $dS_2 = dSn_2$ and $dS_3 = dSn_3$

1. Stress tensor

Equilibrium of the tetraedra leads to

$$dS\underline{T}(M,\underline{n})+dS_1\underline{T}_1(M,-\underline{e}_1)+dS_2\underline{T}_2(M,-\underline{e}_2)+dS_3\underline{T}_3(M,-\underline{e}_3)=\underline{0}$$
$$\Leftrightarrow dS\underline{T}(M,\underline{n})+n_1dS\underline{T}_1(M,-\underline{e}_1)+n_2dS\underline{T}_2(M,-\underline{e}_2)+n_3dS\underline{T}_3(M,-\underline{e}_3)=\underline{0}$$
$$\Rightarrow \underline{T}(M,\underline{n})=n_1\underline{T}_1(M,\underline{e}_1)+n_2\underline{T}_2(M,\underline{e}_2)+n_3\underline{T}_3(M,\underline{e}_3)$$

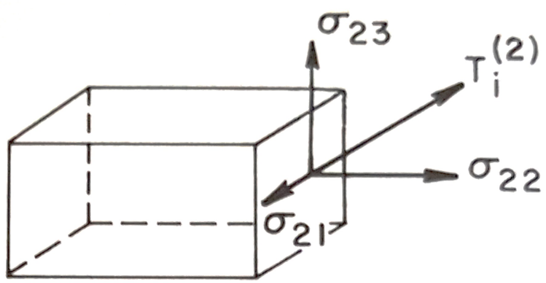
In a condensed form

$$\underline{T}(M,\underline{n})=\left[\{\underline{T}_1(M,\underline{e}_1\}.\{\underline{T}_2(M,\underline{e}_2\}.\{\underline{T}_3(M,\underline{e}_3\}\right].\{\underline{n}\}$$

Or its tensorial expression

$$\underline{T}(M,\underline{n})=[\sigma]\{n\}=\underline{\underline{\sigma}}.\underline{n}$$

1. Stress tensor



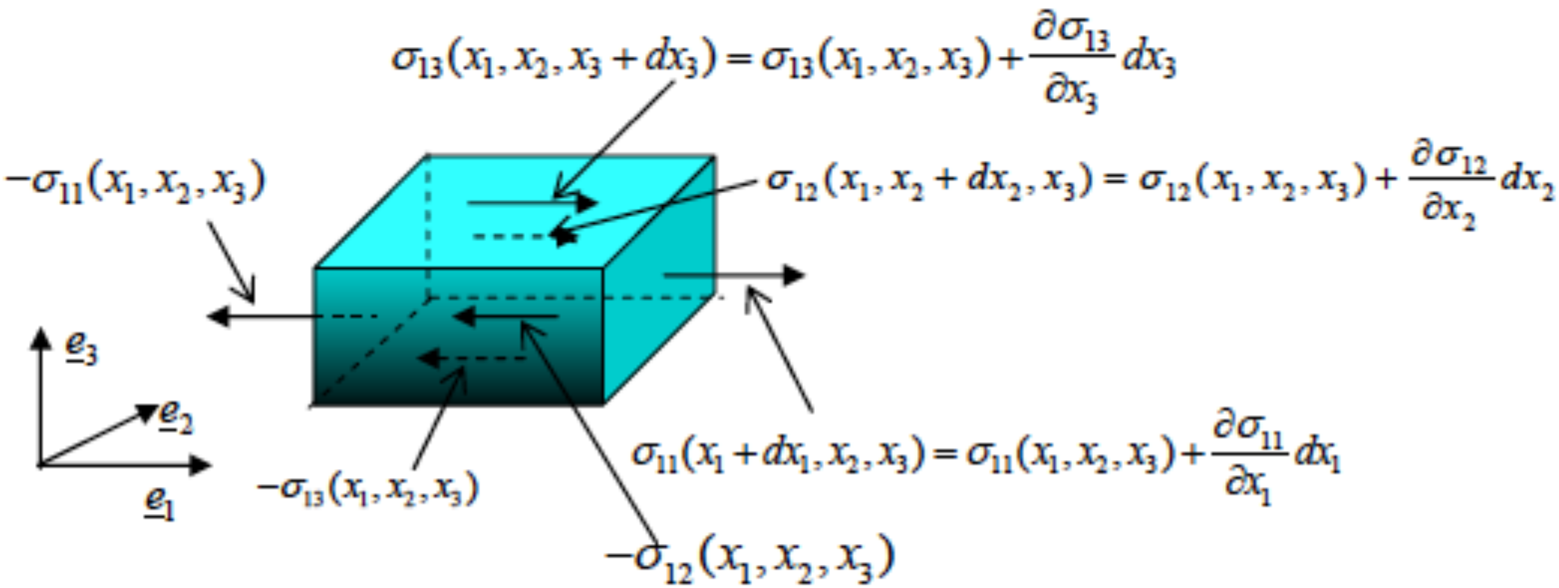
$$T_i^{(2)} = [\sigma_{21}, \sigma_{22}, \sigma_{23}] = [\tau_{yx}, \sigma_y, \tau_{yz}]$$

In the same way for axes 1 and 3 so the stress tensor can be defined as

$$\sigma_{ij} = \begin{bmatrix} T_i^{(1)} \\ T_i^{(2)} \\ T_i^{(3)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

9 components

2. Equilibrium of a small volume



f : a force per unit of volume

The equilibrium a this small volume under the force *f and the contact forces applied by the other part of the body*

2. Equilibrium of a small volume

Projection on $\underline{e}_1$ direction

$$-\sigma_{11}(x_1,x_2,x_3)dx_2dx_3+\sigma_{11}(x_1+dx_1,x_2,x_3)dx_2dx_3$$
$$-\sigma_{12}(x_1,x_2,x_3)dx_1dx_3+\sigma_{12}(x_1,x_2+dx_2,x_3)dx_1dx_3$$
$$-\sigma_{13}(x_1,x_2,x_3)dx_1dx_2+\sigma_{13}(x_1,x_2,x_3+dx_3)dx_1dx_2+f_1dx_1dx_2dx_3=0$$

Which can be rewritten as

$$\frac{\partial\sigma_{11}}{\partial x_1}dV+\frac{\partial\sigma_{12}}{\partial x_2}dV+\frac{\partial\sigma_{13}}{\partial x_3}dV+f_1dV=0$$

or

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+f_1=0$$

On $\underline{e}_2$ and $\underline{e}_3$ directions

$$\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+f_2=0$$

$$\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+f_3=0$$

2. Equilibrium of a small volume

The set of equations can be rewritten as

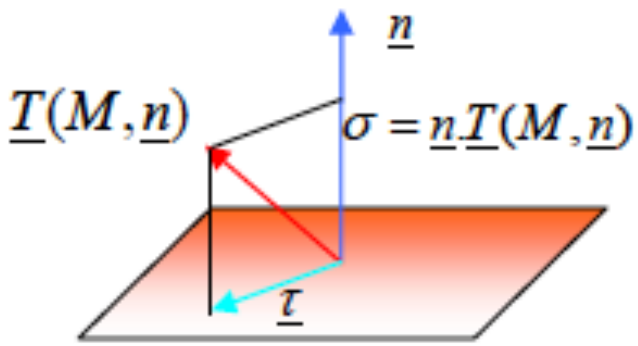
$$\underline{div}\,\underline{\underline{\sigma}} + \underline{f} = \underline{0}$$

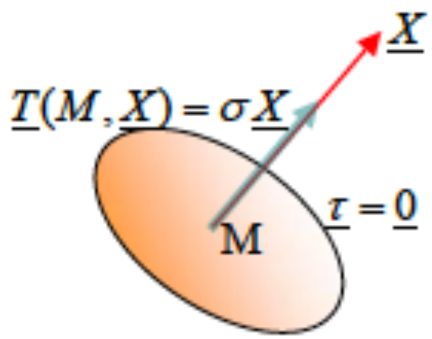
Or

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0$$

3. Principal stresses

We can find cut surfaces oriented in such a way that the shear stress τ vanishes. These directions are called *principal directions.*





$$\underline{T}(M,\underline{n}) = \sigma \underline{n} + \underline{\tau}$$

$$\underline{T}(M,\underline{X}) = \underline{\underline{\sigma}}\,\underline{X} = \sigma \underline{X}$$

3. Principal stresses

From a mathematical point of view, these principal directions can be found be solving the following set of equations

$$\underline{\underline{\sigma}}.\underline{X} = \sigma\underline{X} \quad \text{so} \quad \left(\underline{\underline{\sigma}} - \underline{\underline{I}}\right).\underline{X} = \underline{0}$$

A non trivial solution exist in (e_1, e_2, e_3) space only if

$$\det\left(\underline{\underline{\sigma}} - \underline{\underline{I}}\right).\underline{X} = 0$$

Solutions for the polynomial of 3rd degree exist in σ

$$(\sigma_1, \sigma_2, \sigma_3 \text{ or } \sigma_1 = \sigma_2, \sigma_3 \text{ or } \sigma_1 = \sigma_2 = \sigma_3)$$

3. Principal stresses

In the principal direction space *($\underline{X}_1$, $\underline{X}_2$, $\underline{X}_3$)*, the stress tensor is a diagonal matrix

$$\underline{\underline{\sigma}}_{(\underline{X}_1,\underline{X}_2,\underline{X}_3)} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}$$

Solving $\det\left(\underline{\underline{\sigma}} - \underline{\underline{I}}\right).\underline{X} = 0$ leads to

$$\det\begin{pmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{pmatrix} = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3$$

3. Principal stresses

Where I_1, I_2 and I_3 are the invariants of the stress tensor

$$I_1 = Tr(\underline{\underline{\sigma}}) = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \frac{1}{2}[(Tr\underline{\underline{\sigma}})^2 - Tr(\underline{\underline{\sigma}}^2)] = \sigma_{11}\sigma_{33} + \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 = \sigma_1\sigma_3 + \sigma_1\sigma_2 + \sigma_2\sigma_3$$

$$I_3 = \det \underline{\underline{\sigma}}$$

3. Principal stresses

In the principal space (M; X1, X 2 , X 3) , the components of the stress vector acting on the surface with *n* vector are

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} . \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{Bmatrix}$$

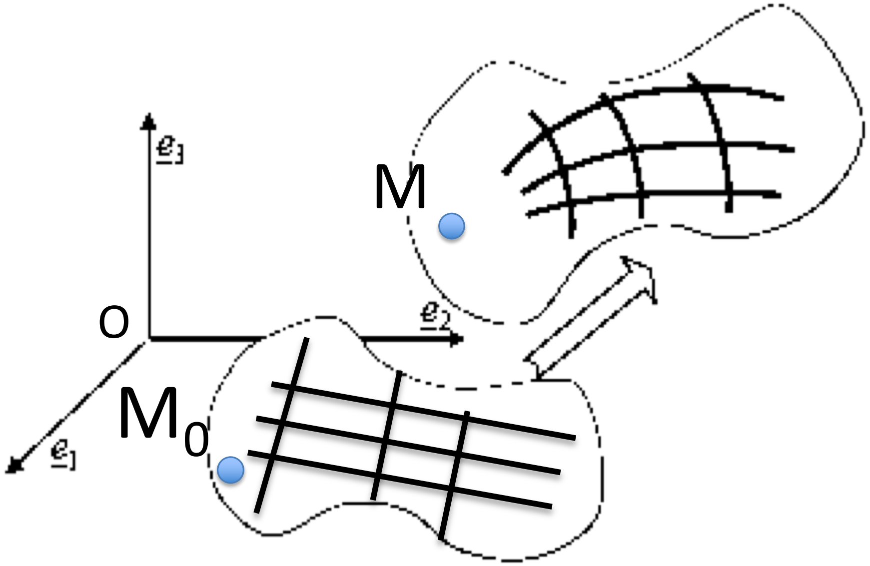
where n_1, n_2 and n_3 are the component of the unity vector *n* given by $n_1^2 + n_2^2 + n_3^2 = 1$

Finally, one obtain

$$\frac{T_1^2}{n_1^2} + \frac{T_2^2}{n_2^2} + \frac{T_3^2}{n_3^2} = 1$$

Lame ellipsoid

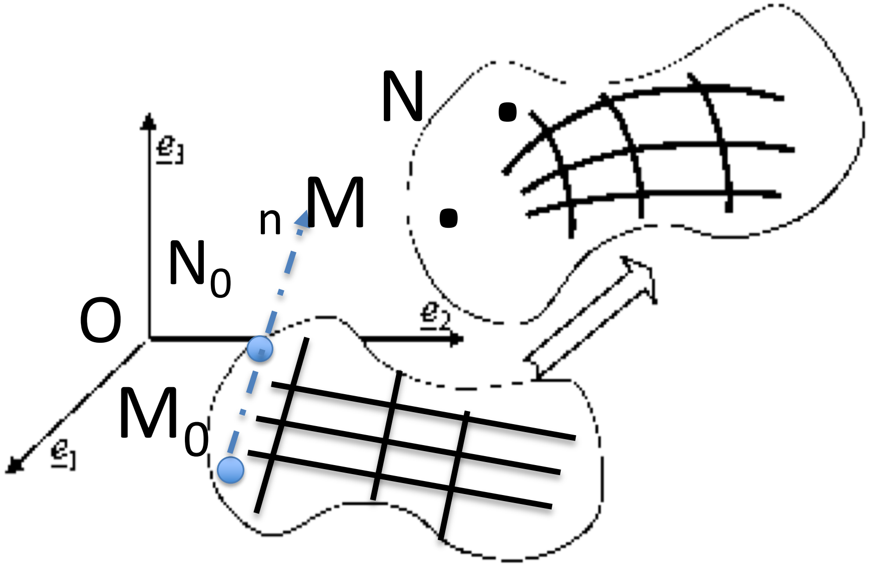
4. Strain tensor (roughly introduced)



During loading, a solid particle in point M_0 with initial coordinates X in moving and occupying a new position in point M with coordinates x. The displacement vector of point M0(X) is given by

$$\underline{\underline{M_0M}} = \underline{x} - \underline{X} = \underline{u}(X_1, X_2, X_3) = u_1(X_1, X_2, X_3)\underline{e}_1 + u_2(X_1, X_2, X_3)\underline{e}_2 + u_3(X_1, X_2, X_3)\underline{e}_3$$

4. Strain tensor

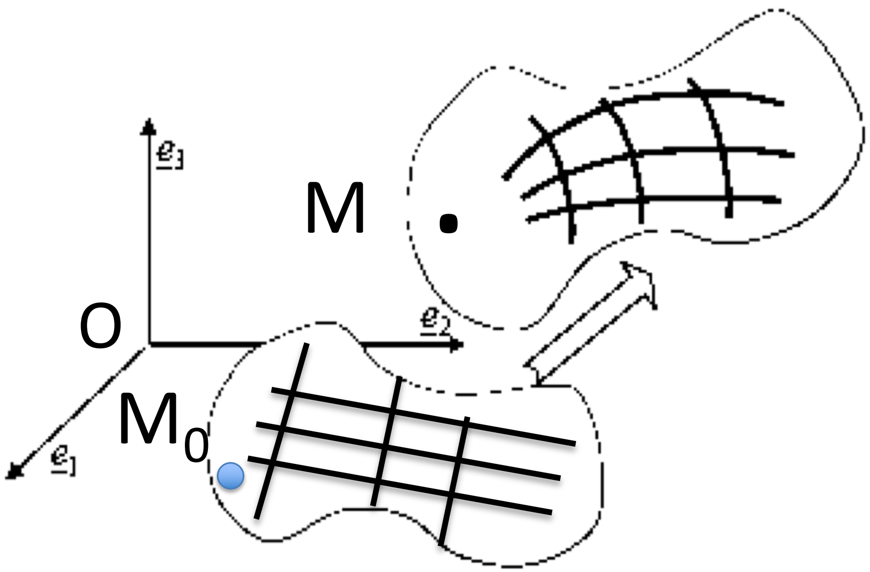


The solid is deformed under loading and points M_0 and N_0 are moving to M et N respectively

The unit increase (or decrease) in size is given by

$$\varepsilon(M_0, \underline{n}) = \lim_{N_0 \to M_0} \frac{\left\|\underline{MN}\right\| - \left\|\underline{M_0N_0}\right\|}{\left\|\underline{M_0N_0}\right\|}$$

4. Strain tensor



The displacement of a particle occupying the position from M_0 at time t_0 to M at time t is given by

$$\underline{OM} = \underline{OM_0} + \underline{u} \quad \text{or} \quad \underline{x} = \underline{X} + \underline{u}$$

where $\underline{u}$ is the displacement vector between t_0 and t.

4. Strain tensor

The displacement of a small vector *dX* instead of vector *X*

Therefore, we can write

$$\underline{dx} = \underline{dX} + \underline{du}$$

The displacement vector *du* which dependant on *X* , is given by

$$\underline{du} = \frac{\partial u_i}{\partial X_j} dX_j \underline{e}_i = \underline{\underline{\nabla u}}.\underline{dX}$$

Or by

$$\underline{dx} = \underline{dX} + \underline{du} = \underline{\underline{F}}.\underline{dX}$$

Where *F* is called the gradiant of the transformation

$$\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{\nabla u}}$$

and *I* is the unit matrix

4. Strain tensor

The gradient of the displacement vector can be decomposed in a symmetric and screw symmetric parts (small perturbations)

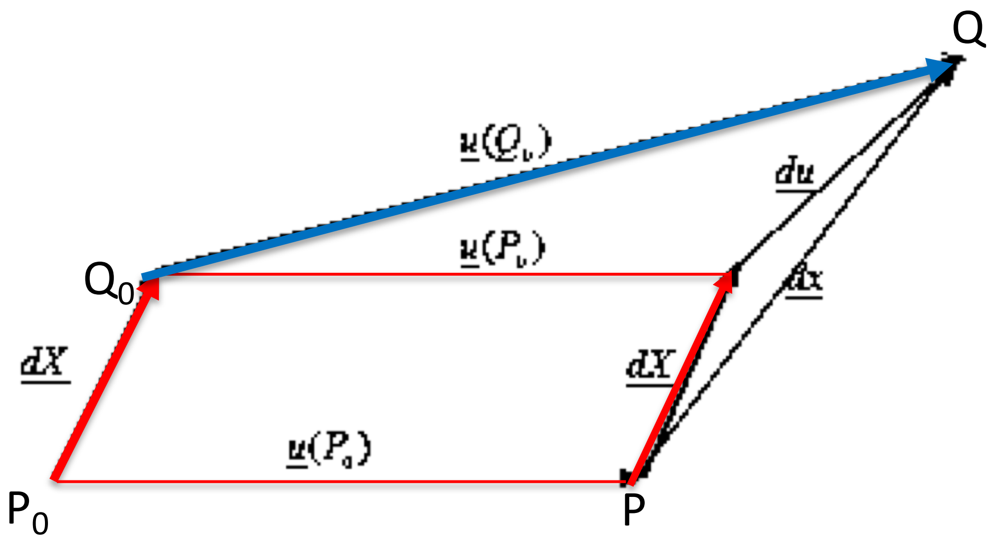
$$\underline{\underline{\nabla \underline{u}}} = \underbrace{\frac{1}{2}\left[\underline{\underline{\nabla \underline{u}}} + {}^t\underline{\underline{\nabla \underline{u}}}\right]}_{\underline{\underline{\varepsilon}}} + \underbrace{\frac{1}{2}\left[\underline{\underline{\nabla \underline{u}}} - {}^t\underline{\underline{\nabla \underline{u}}}\right]}_{\underline{\underline{\Omega}}}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$[\varepsilon] = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

4. Strain tensor



$$\underline{u}(Q_0) = \underbrace{\underline{u}(P_0)}_{Translation} + \underbrace{\underline{\underline{\Omega}}\,\underline{dX}}_{Rotation} + \underbrace{\underline{\underline{\varepsilon}}.\underline{dX}}_{Deformation}$$

For rigid bodies: $\underbrace{\underline{\underline{\Omega}}\,\underline{dX}}_{Rotation} = \underbrace{\underline{\underline{\varepsilon}}.\underline{dX}}_{Deformation} = 0$ so $\underline{u}(Q_0) = \underbrace{\underline{u}(P_0)}_{Translation}$

4. Strain tensor

Form small perturbations

$$\underline{u}(M,\underline{n}) = \underline{\underline{\varepsilon}}(M).\underline{n}$$

$$\underline{u}_1(M,\underline{e}_1) = \varepsilon_{11}\underline{e}_1 + \varepsilon_{12}\underline{e}_2 + \varepsilon_{13}\underline{e}_3$$

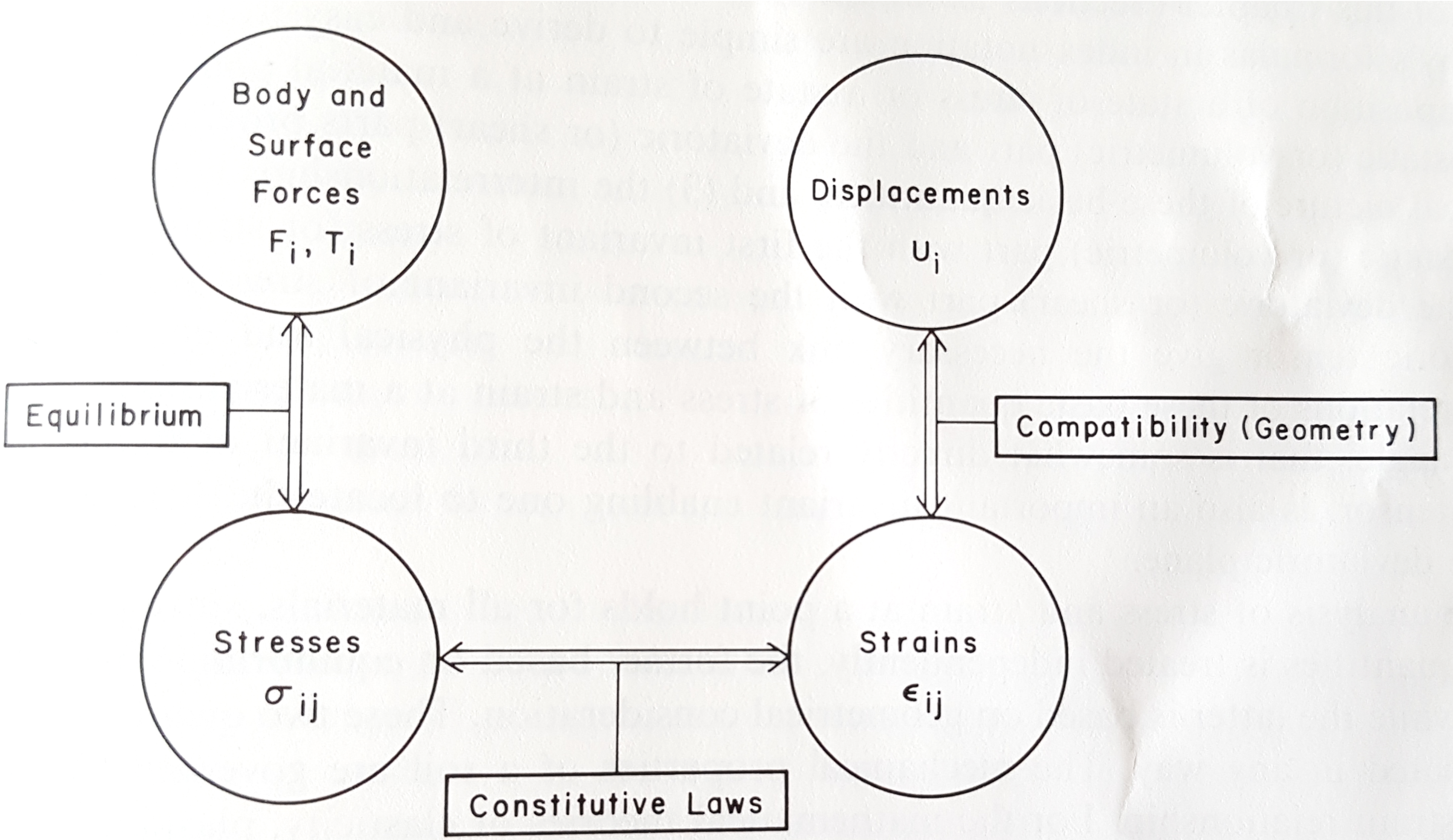
$$\underline{u}_2(M,\underline{e}_2) = \varepsilon_{21}\underline{e}_1 + \varepsilon_{22}\underline{e}_2 + \varepsilon_{23}\underline{e}_3$$

$$\underline{u}_3(M,\underline{e}_3) = \varepsilon_{31}\underline{e}_1 + \varepsilon_{32}\underline{e}_2 + \varepsilon_{31}\underline{e}_3$$

Similarly to principal stress components, we can obtain principal strain components

$$[\varepsilon]_{(\underline{E}_1,\underline{E}_2,\underline{E}_3)} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

Basic concepts of CM



ELASTICITY

1. Linear elasticity

2. Nonlinear elasticity

Linear elasticity

- Features
  - In the elastic domain, a material recover its initial state, after being loaded and deformed, when the external loading is stopped.
  - The stress state is only depend on the strain state (and *vice versa*)

Free energy and constitutive relations

The free energy per unit volume is defined as *Ψ(ε)*, where *ε* is the (macroscopic) strain .

Ψ(ε) depends only on the strain state and is called the strain energy. This holds for non dissipative materials

From *Ψ(ε)* we calculate the stress *σ* from the constitutive equations:

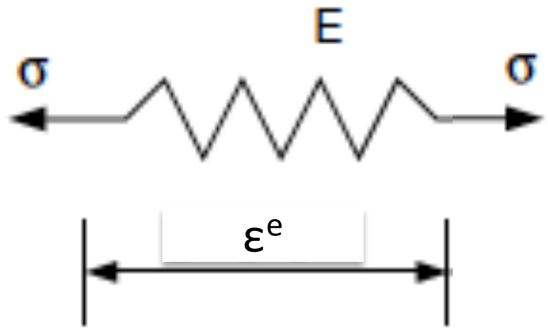
$$\sigma = \frac{\partial \Psi(\varepsilon)}{\partial \varepsilon}$$

Linear elasticity

1D case (Hooke's law)

The simplest choice of the energy that provide the constitutive behaviour is

$$\Psi(\varepsilon) = \frac{1}{2} E \varepsilon^2$$



From which we obtain the stress

$$\sigma = \frac{\partial \Psi(\varepsilon)}{\partial \varepsilon} = E\varepsilon$$

Linear elasticity

- Isotropic linear elasticity (3D)

$E_{11} = E_{22} = E_{33} = E$

$\nu_{ij} = \nu$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \times \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix}$$

$$\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

Linear elasticity

- Isotropic linear elasticity

Inverted the previous equation leads to

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2G \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}$$

G, the shear modulus is expressed by $G = E/2(1 + \nu)$.

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Formules de conversion

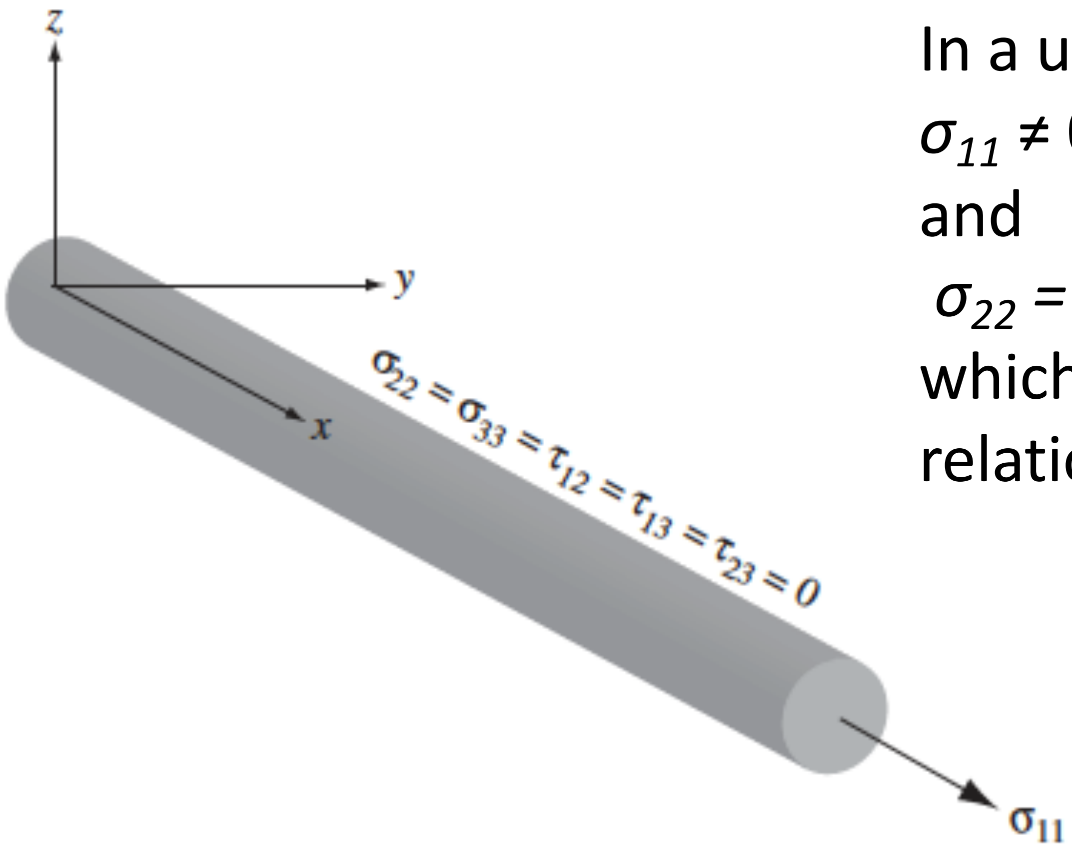
Les propriétés élastiques des matériaux homogènes, isotropes et linéaires sont déterminées de manière unique par deux modules quelconques parmi ceux-ci. Ainsi, on peut calculer chacun à partir de deux d'entre eux en utilisant ces formules.

https://fr.wikipedia.org/wiki/Module_d%27élasticité

	(λ, G)	(E, G)	(K, λ)	(K, G)	(λ, ν)	(G, ν)	(E, ν)	(K, ν)	(K, E)	(M, G)
$K =$	$\lambda + \frac{2G}{3}$	$\frac{EG}{3(3G-E)}$			$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$			$M - \frac{4G}{3}$
$E =$	$\frac{G(3\lambda+2G)}{\lambda+G}$		$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{9KG}{3K+G}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2G(1+\nu)$		$3K(1-2\nu)$		$\frac{G(3M-4G)}{M-G}$
$\lambda =$		$\frac{G(E-2G)}{3G-E}$		$K - \frac{2G}{3}$		$\frac{2G\nu}{1-2\nu}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{3K\nu}{1+\nu}$	$\frac{3K(3K-E)}{9K-E}$	$M - 2G$
$G =$			$\frac{3(K-\lambda)}{2}$		$\frac{\lambda(1-2\nu)}{2\nu}$		$\frac{E}{2(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$\frac{3KE}{9K-E}$	
$\nu =$	$\frac{\lambda}{2(\lambda+G)}$	$\frac{E}{2G} - 1$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2G}{2(3K+G)}$					$\frac{3K-E}{6K}$	$\frac{M-2G}{2M-2G}$
$M =$	$\lambda + 2G$	$\frac{G(4G-E)}{3G-E}$	$3K - 2\lambda$	$K + \frac{4G}{3}$	$\frac{\lambda(1-\nu)}{\nu}$	$\frac{2G(1-\nu)}{1-2\nu}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\frac{3K(1-\nu)}{1+\nu}$	$\frac{3K(3K+E)}{9K-E}$	

Linear elasticity

- Uniaxial Stress Condition



Rebar under uniaxial stress condition

In a uniaxial stress condition we have:

$\sigma_{11} \neq 0$

and

$\sigma_{22} = \sigma_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$

which implies that the previous relation becomes

Linear elasticity

- Uniaxial Stress Condition

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2G \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Linear elasticity

- Uniaxial Stress Condition

This reduces to two equations

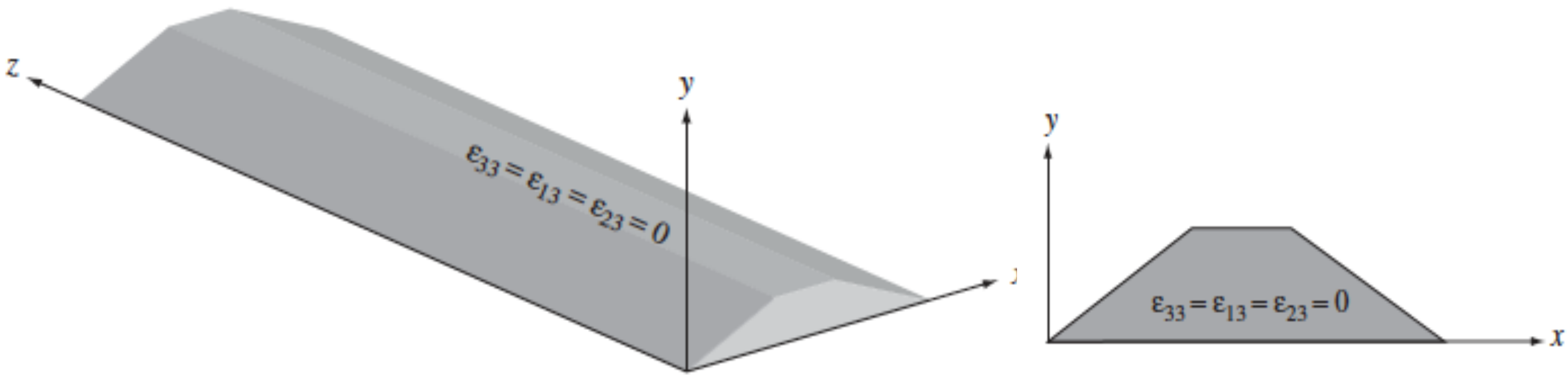
$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} \quad \text{and} \quad \varepsilon_{22} = \varepsilon_{33} = \frac{-\nu}{E}\sigma_{11}$$

So finally $\nu = \frac{-\varepsilon_{33}}{\varepsilon_{11}}$ or $\varepsilon_{33} = -\nu\varepsilon_{11}$

The axial stress causes the steel rebar to extend in the axial direction, the rebar becomes slimmer (negative ε_{33}), due to Poisson's effect.

Plane Strain Condition

Structures that are very long in one dimension while having a uniform cross section with finite dimensions



Soil embankment in plane strain conditions

Plane Strain Condition

The strains along the z-axis are assumed to be nil :

$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \times \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 0 \\ \varepsilon_{12} \\ 0 \\ 0 \end{Bmatrix}$$

Plane Strain Condition

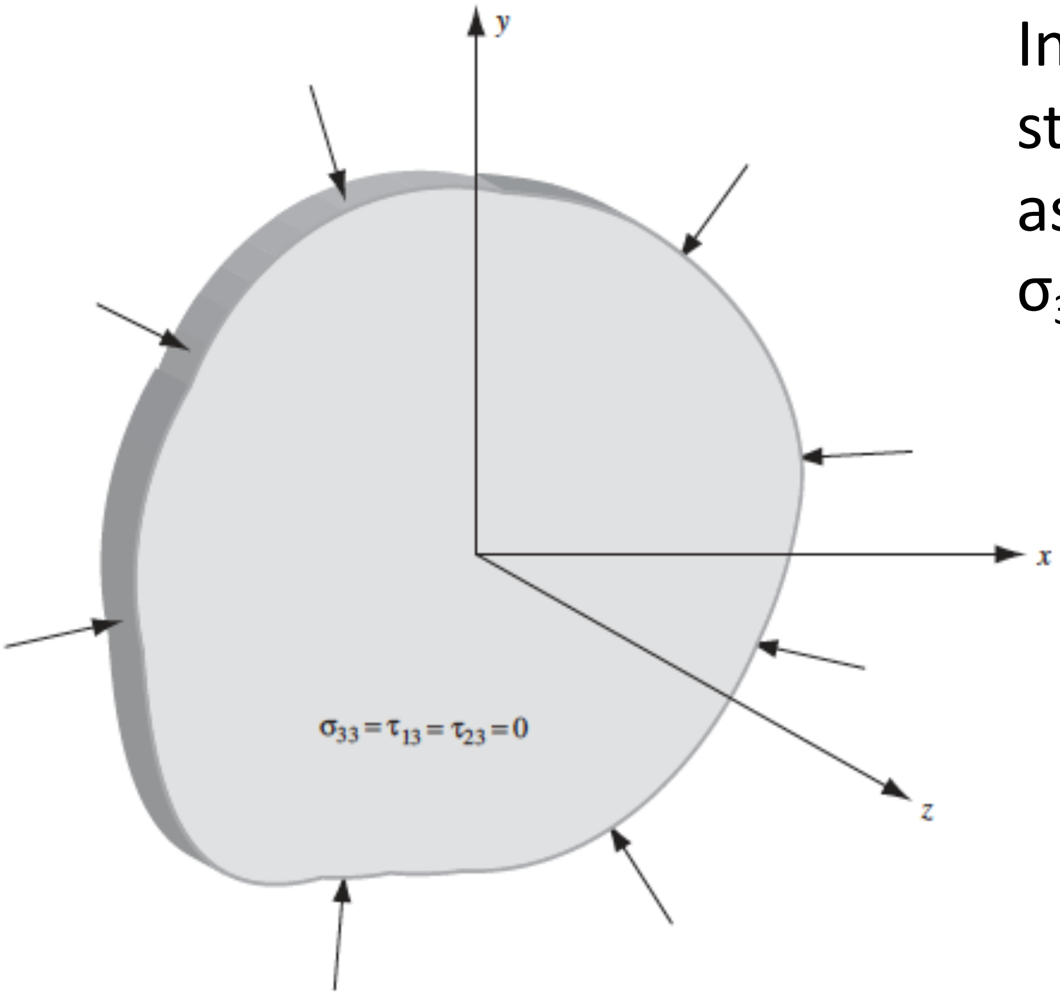
Which can be reduced to

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}$$

By inverting

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

Plane Stress Condition



In the plane stress condition the stresses in the z-direction are assumed negligible

$\sigma_{33} = \tau_{13} = \tau_{23} = 0$

Linear elasticity

Plane Stress Condition

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2G \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ \tau_{12} \\ 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

Plane Stress Condition

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix}\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

Inverting we obtain

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}$$

To summarize

Plane Strain Condition

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

Plane Stress Condition

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

To summarize

Plane Strain Condition

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}$$

Plane Stress Condition

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}$$

Linear elasticity

- <u>Constants</u>
  - Young modulus *E* and Poisson's coefficient
  - Or shear modulus *G* and the isotropic bulk modulus *K*
  - Or Lamé coefficients

- <u>Limitation</u>
  - Does not consider the loading path and the history of loading

Nonlinear elasticity

Nonlinear elasticity

- Features
  - If the model is derived from a potential: hyperelasticity
  - If it does not derived from a thermodynamic potentiel : hypoelasticity
  - Stress increment depends not only from the strain's increment but also from the actual stress
- Constitutive relation

$$\dot{\sigma}_{ij} = C_{ijkl}(\sigma_{mn})\dot{\varepsilon}_{kl} \quad \text{and} \quad \dot{\sigma}_{ij} = C_{ijkl}(\varepsilon_{mn})\dot{\varepsilon}_{kl}$$

$$\dot{\varepsilon}_{ij} = D_{ijkl}(\sigma_{mn})\dot{\sigma}_{kl} \quad \text{and} \quad \dot{\varepsilon}_{ij} = D_{ijkl}(\varepsilon_{mn})\dot{\sigma}_{kl}$$

Nonlinear hyperelasticity

- Features
  - There is no intrinsic dissipation
  - The Model is characterized by the knowledge of the free energy (W_{ij}) or the complementary energy density function (Ω_{ij}) such as $W + \Omega = \sigma_{ij}\varepsilon_{ij}$

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{kl})}{\partial \varepsilon_{ij}} \quad \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad d\sigma_{ij} = \frac{\partial^2 W}{\partial \varepsilon_{ij}\,\partial \varepsilon_{kl}} d\varepsilon_{kl} = H_{ijkl}\, d\varepsilon_{kl}$$

$$\varepsilon_{ij} = \frac{\partial \Omega(\sigma_{kl})}{\partial \sigma_{ij}} \quad \varepsilon_{ij} = D_{ijkl}\,\sigma_{kl} \quad d\varepsilon_{ij} = \frac{\partial^2 W}{\partial \sigma_{ij}\,\partial \sigma_{kl}} d\sigma_{kl} = H'_{ijkl}\, d\sigma_{kl}$$

Nonlinear hyperelasticity

$$[H] = \begin{bmatrix} \frac{\partial^2 W}{\partial \epsilon_{11}^2} & \frac{\partial^2 W}{\partial \epsilon_{11}\partial \epsilon_{22}} & \frac{\partial^2 W}{\partial \epsilon_{11}\partial \epsilon_{33}} & \frac{\partial^2 W}{\partial \epsilon_{11}\partial \gamma_{12}} & \frac{\partial^2 W}{\partial \epsilon_{11}\partial \gamma_{23}} & \frac{\partial^2 W}{\partial \epsilon_{11}\partial \gamma_{31}} \\ & \frac{\partial^2 W}{\partial \epsilon_{22}^2} & \frac{\partial^2 W}{\partial \epsilon_{22}\partial \epsilon_{33}} & \frac{\partial^2 W}{\partial \epsilon_{22}\partial \gamma_{12}} & \frac{\partial^2 W}{\partial \epsilon_{22}\partial \gamma_{23}} & \frac{\partial^2 W}{\partial \epsilon_{22}\partial \gamma_{31}} \\ & & \frac{\partial^2 W}{\partial \epsilon_{33}^2} & \frac{\partial^2 W}{\partial \epsilon_{33}\partial \gamma_{12}} & \frac{\partial^2 W}{\partial \epsilon_{33}\partial \gamma_{23}} & \frac{\partial^2 W}{\partial \epsilon_{33}\partial \gamma_{31}} \\ & & & \frac{\partial^2 W}{\partial \gamma_{12}^2} & \frac{\partial^2 W}{\partial \gamma_{12}\partial \gamma_{23}} & \frac{\partial^2 W}{\partial \gamma_{12}\partial \gamma_{31}} \\ & \text{Symmetric} & & & \frac{\partial^2 W}{\partial \gamma_{23}^2} & \frac{\partial^2 W}{\partial \gamma_{23}\partial \gamma_{31}} \\ & & & & & \frac{\partial^2 W}{\partial \gamma_{31}^2} \end{bmatrix}$$

Examples of hyperelastic models

Nonlinear hyperelasticity

Boyce (1980)	$K = \dfrac{K_1 p^{(1-n)}}{\left(1-(1-n)\dfrac{K_1}{6G_1}\dfrac{q^2}{p^2}\right)}$ $G = G_1 p^{(1-n)}$	K_1, G_1 et n
Loret (1981)	v = Constante ou K/G = Constante $E = E_0 P_a \left[\left(\dfrac{p}{p_a}\right)^2 \left(1+\dfrac{K}{3G}\dfrac{q^2}{p^2}\right)\right]^{\frac{1-n}{2}}$ ou $G = G_0 P_a \left(\dfrac{p}{p_a}\right)^{1-n}$ $K = K_0 P_a \left(\dfrac{p}{p_a}\right)^{1-n} \dfrac{1}{1-\dfrac{K_0}{6G_0}(1-n)\dfrac{q^2}{p^2}}$	v, E_0 et n ou G_0, K_0 et n

Nonlinear hyperelasticity

Mroz et Norris (1982)	$e_{ij}^{e} = \frac{s_{ij}}{2G}$ $\varepsilon_{v}^{e} = \frac{\kappa}{1+e_0} \ln\left(\frac{p}{p_{co}}\right) - \frac{s_{ij}s_{ij}}{4G^2}\frac{\partial G}{\partial p}$	e_0, κ et G
Chen et Baladi (1985)	$K = \frac{K_i}{1-K_1}\left[1 - K_1 \exp(-K_2 I_1)\right]$ $G = \frac{G_i}{1-G_1}\left[1 - G_1 \exp\left(-G_2\sqrt{J_2}\right)\right]$	K_i, K_1, K_2, G_i, G_1 et G_2
Lade et Nelson (1987)	$E = Mp_a\left[\left(\frac{I_1}{p_a}\right)^2 + 6\left(\frac{1+\nu}{1-2\nu}\right)\frac{J_2}{p_a^2}\right]^{\lambda}$ ν = Constante	ν, M p_a et λ

Nonlinear hyperelasticity

Cambou et Jafari (1988)	$G = G_0\left[\frac{I_1}{3p_a}\right]^n$ $K = K_0^e\left[\frac{I_1}{3p_a}\right]^n \frac{4G_0I_1^2}{4G_0I_1^2 - 9nK_0^eS_{ij}S_{ij}}$	G_0, K_0^e, p_a et n
Molenkamp (1988)	$K = \frac{1}{3}\frac{p_a}{AP}\left(\frac{I_1}{\sqrt{3}p_a}\right)^{1-P}$ $G = \frac{Rp_a}{3AP}\left(\frac{I_1}{\sqrt{3}p_a}\right)^{1-P}$	A, P et R
Huang et Gibson (1993)	$K = \frac{K_0}{1-6\left(\frac{1-\nu_0}{1-2\nu_0}\right)K_1V_s}$ $G = \frac{G_0}{1+24(1-\nu_0)G_1V_s}$	G_0, K_0 ν_0 K_1 G_1 et V_s

Nonlinear hyperelasticity

Houlsby model

The elastic strain energy φ is written as a function of volumetric strain and shear strain: $\varphi = \varphi\ (\varepsilon_v, \varepsilon_s)$

$$p = \frac{\partial \varphi}{\partial \varepsilon_v}, q = \frac{\partial \varphi}{\partial \varepsilon_s}$$

The incremental stiffness matrix can be expressed as:

$$\begin{bmatrix} dp \\ dq \end{bmatrix} = \begin{bmatrix} K & J \\ J & 3G \end{bmatrix} \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_s \end{bmatrix}$$

$$K = \frac{\partial p}{\partial \varepsilon_v} = \frac{\partial^2 \varphi}{\partial {\varepsilon_v}^2}, 3G = \frac{\partial q}{\partial \varepsilon_s} = \frac{\partial^2 \varphi}{\partial {\varepsilon_s}^2}, J = \frac{\partial p}{\partial \varepsilon_s} = \frac{\partial q}{\partial \varepsilon_v} = \frac{\partial^2 \varphi}{\partial \varepsilon_s \partial \varepsilon_v}$$

Houlsby model

According to Houlsby [15] and Einav [16], though φ is an isotropic function of strains, the soil behaves incrementally like anisotropic way when the value of off-diagonal terms J is non-zero, in which situation the stress-induced anisotropy shows up. The free energy expression could be written as:

$$\varphi = p_a\left(\frac{k}{2}{\varepsilon_v}^2 + \frac{3g}{2}{\varepsilon_s}^2\right)$$

with

k : bulk stiffness factor, g shear stiffness factor (dimensionless constants for linear elasticity)

p_a for the reference stress.

Nonlinear hyperelasticity

Houlsby model

For non-linear elasticity, the proposed hyperelastic potential can be written as:

$$\varphi = \frac{Pa}{k(2-n)}\left[k \cdot v_0 \cdot (1-n)\right]^{\frac{2-n}{1-n}}$$

with triaxial formulation:

$$v_0^2 = {\varepsilon_v^*}^2 + \frac{3g \cdot \varepsilon_s^2}{k(1-n)}$$

$$\varepsilon_v^* = \varepsilon_v + \frac{1}{k(1-n)}$$

while with general stress formulation:

$$v_0^2 = \left[\varepsilon_{ii} + \frac{1}{k(1-n)}\right]\left[\varepsilon_{jj} + \frac{1}{k(1-n)}\right] + \frac{2g \cdot e_{ij}e_{ij}}{k(1-n)}$$

n stands for pressure exponent, and has a significant influence on the effect of induced anisotropy within the range between zero and one.

Nonlinear hyperelasticity

Houlsby model

As a result, the formulas of bulk modulus, shear modulus and off-diagonal terms in stiffness matrix are expressed as follows.

$$K = p_a[k(1-n)]^{\frac{1}{1-n}} \cdot \left\{\frac{n}{1-n} \cdot \left(\varepsilon_v + \frac{1}{k(1-n)}\right)^2 \cdot \left[\left(\varepsilon_v + \frac{1}{k(1-n)}\right)^2 + \frac{3g\varepsilon_s^2}{k(1-n)}\right]^{\frac{3n-2}{2-2n}} + \left[\left(\varepsilon_v + \frac{1}{k(1-n)}\right)^2 + \frac{3g\varepsilon_s^2}{k(1-n)}\right]^{\frac{n}{2-2n}}\right\}$$

$$3G = p_a[k(1-n)]^{\frac{n}{1-n}} \cdot 3g \cdot \left\{\frac{n \cdot 3g \cdot \varepsilon_s^2}{k(1-n)^2} \cdot \left[\left(\varepsilon_v + \frac{1}{k(1-n)}\right)^2 + \frac{3g\varepsilon_s^2}{k(1-n)}\right]^{\frac{3n-2}{2-2n}} + \left[\left(\varepsilon_v + \frac{1}{k(1-n)}\right)^2 + \frac{3g\varepsilon_s^2}{k(1-n)}\right]^{\frac{n}{2-2n}}\right\}$$

$$J = p_a[k(1-n)]^{\frac{n}{1-n}} \cdot \frac{3g \cdot n \cdot \varepsilon_s}{1-n} \cdot \left[\varepsilon_v + \frac{1}{k(1-n)}\right] \cdot \left[\left(\varepsilon_v + \frac{1}{k(1-n)}\right)^2 + \frac{3g\varepsilon_s^2}{k(1-n)}\right]^{\frac{3n-2}{2-2n}}$$

Hypoelastic models

Nonlinear hypoelasticity

- <u>features</u>
  - Does not derive from a thermodynamic potential
  - Stress increment depends not only from the strain's increment but also from the actual stress

- <u>Équations</u>

$\dot{\sigma}_{ij} = C_{ijkl}(\sigma_{mn})\dot{\varepsilon}_{kl}$ and $\dot{\sigma}_{ij} = C_{ijkl}(\varepsilon_{mn})\dot{\varepsilon}_{kl}$

$\dot{\varepsilon}_{ij} = D_{ijkl}(\sigma_{mn})\dot{\sigma}_{kl}$ and $\dot{\varepsilon}_{ij} = D_{ijkl}(\varepsilon_{mn})\dot{\sigma}_{kl}$

Nonlinear hypoelasticity

Ramberg et Osgood (1943)	$\varepsilon = \frac{\sigma}{E} + K_1\left(\frac{\sigma}{E}\right)^n$	E, K_1 et n
Rivlin et Erickseen (1955)	$\dot{\sigma}_{ij} = \alpha_0\delta_{ij} + \alpha_1\dot{\varepsilon}_{ij} + \alpha_2\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} + \alpha_3\sigma_{ij} + \alpha_4\sigma_{ik}\sigma_{kj} + \alpha_5\left(\dot{\varepsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\varepsilon}_{kj}\right) + \alpha_6\left(\dot{\varepsilon}_{ik}\dot{\varepsilon}_{km}\sigma_{mj} + \sigma_{ik}\dot{\varepsilon}_{km}\dot{\varepsilon}_{mj}\right) + \alpha_7\left(\dot{\varepsilon}_{ik}\sigma_{km}\sigma_{mj} + \sigma_{ik}\sigma_{km}\dot{\varepsilon}_{mj}\right) + \alpha_8\left(\dot{\varepsilon}_{ik}\dot{\varepsilon}_{km}\sigma_{mn}\sigma_{nj} + \sigma_{ik}\sigma_{km}\dot{\varepsilon}_{mn}\dot{\varepsilon}_{nj}\right)$	α_0, α_1, α_2, α_3, α_4, α_5, α_6, α_6, α_7 et α_8
Hansen (1963)	$\sigma = \left(\frac{\varepsilon}{a + b\varepsilon}\right)^{1/2}$	a et b
Kondner (1963)	$\sigma = \frac{\varepsilon}{a + b\varepsilon}$	a et b
Janbu (1963)	$E_i = K_h p_a\left(\frac{\sigma_3}{p_a}\right)^n$	K_h et n

Nonlinear hypoelasticity

Richardson et Whitman (1963) (d'après Ramberg et Osgood, 1943)	K = Constante et $G = \dfrac{G_{max}}{1+\alpha\left(\dfrac{\tau}{G_{max}\gamma}\right)^{R-1}}$	K, α, G_{max} et R
Holubec (1968)	Isotrope : $\varepsilon_1^e = a\cdot(p)^{1/2}$ Anisotrope : $\varepsilon_1^e = c\cdot(p)^{2/3}$	a et c
Roscoe et Burland (1968)	$K = \dfrac{1+e_0}{\kappa}p$ et $G = \infty$	e_0 et κ
Domaschuk et Wade (1969)	$K_t(\sigma_m) = K_0 + m\sigma_m$ $G_t(\sigma_m, \tau_{oct}) = G_0(1-b\tau_{oct})^2$	K_0, m, G_0 et b

Nonlinear hypoelasticity

Duncan et Chang (1970)	loading $E_t = \left[1 - \frac{R_f(1-\sin\phi)(\sigma_1-\sigma_3)}{2(c\cos\phi + \sigma_3\sin\phi)}\right]^2 k_h p_a \left(\frac{\sigma_3}{p_a}\right)^n$ ν = Constante Unloading - reloading $E_{ur} = K_{ur} p_a \left(\frac{\sigma_3}{p_a}\right)^n$	ν, c, ϕ, R_f, k_h, n et K_{ur}
Richart, Hall et Woods (1970)	$K = \frac{1+e_0}{\kappa} p_a \left(\frac{p}{p_a}\right)^{1/2}$ $G = G_0 \frac{(2,973-e_0)^2}{1+e_0} p_a \left(\frac{p}{p_a}\right)^{1/2}$	e_0, κ et G_0
Nelson et Baron (1971)	loading $K = K_0 + K_1\varepsilon_v + K_2\varepsilon_v^2$ Unloading - reloading $G = G_0 + \alpha_1 p + \alpha_2\sqrt{J_2}$	K_0, K_1 et K_2 G_0, α_1 et α_2
Desai (1971, 1972)	$G = \frac{\partial\tau}{\partial\gamma} = a_1 + 2a_2\gamma + 3a_3\gamma^2 + \ldots + na_n\gamma^{n-1} + \ldots$	a_1, a_2, a_3,...,et a_n

Nonlinear hypoelasticity

Vermeer (1978)	$G = G_0\left(\frac{p}{p_0}\right)^{1-\beta}$ et $\nu =$ Constante	G_0, p_0, β et ν
Baladi et Rohani (1979)	$K = K_i + K_1 p'$ $G =$ Constante	G, K_i et K_1
Bazant et Tsubaki (1980)	$\frac{1}{K} = \frac{1}{K_0} + Q_1$ et $\frac{1}{G} = \frac{1}{G_0} + 2P_1$	K_0, G_0, Q_1 et P_1

PERFECT PLASTICITY

Perfect plasticity

Assumption

The material is considered as dry or saturated. In the latter case, the effective stress is defined by the Terzaghi's relation.

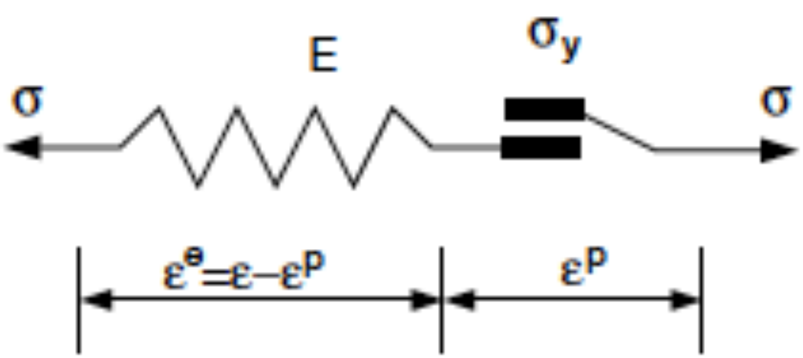
$\sigma'_{ij} = (\sigma_{ij} - u)$ where u is the pore water pressure.

In Soil Mechanics, constitutive models are written for effective stresses.

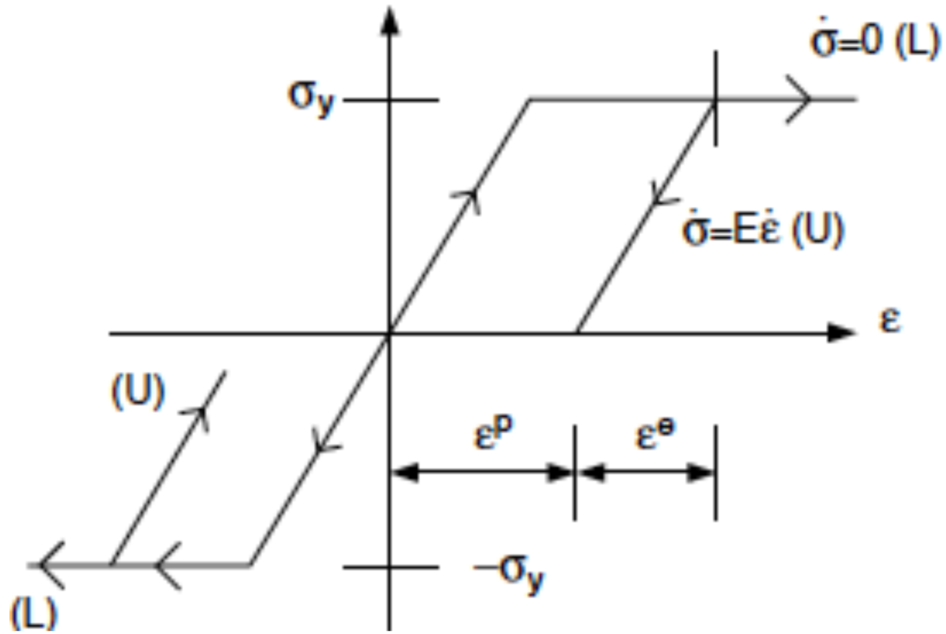
Perfect plasticity

Thermodynamic basis — Yield criterion

The frictional-plastic slider is inactive as long as $|\sigma| < \sigma_y$, where σ_y is the yield stress.



Prototype model for elastic-(perfectly)-plastic material



Stress-strain relationship.

(L: loading , U: unloading)

Perfect plasticity

As the single internal variable, we take the plastic strain ε^p, and the expression for the free energy is chosen as

$$\Psi = \frac{1}{2}E(\epsilon^{\mathrm{e}})^2 = \frac{1}{2}E(\epsilon - \epsilon^{\mathrm{p}})^2$$

where $\varepsilon^e = \varepsilon - \varepsilon^p$ is the elastic strain of the Hookean spring with modulus of elasticity E.

Perfect plasticity

We then obtain the constitutive equation for the stress as

$$\sigma = \frac{\partial \Psi}{\partial \epsilon} = E(\epsilon - \epsilon^{\mathrm{p}})$$

and for the dissipative stress, that is conjugated to ε^{p}, as

$$\sigma^{\mathrm{p}} = -\frac{\partial \Psi}{\partial \epsilon^{\mathrm{p}}} = E(\epsilon - \epsilon^{\mathrm{p}}) \equiv \sigma$$

The yield criterion is $\Phi = 0$, where Φ is chosen as

$$\Phi(\sigma) = |\sigma| - \sigma_{\mathbf{y}}$$

Perfect plasticity

Remark:

Since the magnitude of stress can never exceed the yield stress (in this simple prototype model), it follows that the admissible stress range is defined as those stresses for which $\Phi \leq 0$.

$$\Phi(\sigma) = |\sigma| - \sigma_y$$

Plastic flow rule

It is assumed that no plastic strain will be produced when $\Phi < 0$, i.e. when $|\sigma| < \sigma_y$.

The material response is then elastic and $|\sigma| < \sigma_y$ thus defines the elastic stress range.

However, when $\Phi = 0$ plastic strain may be produced. The constitutive rate equation for ε^p is then postulated as the associative flow rule

$$\dot{\varepsilon}^{p} = \lambda \frac{\partial \Phi}{\partial \sigma} = \lambda \frac{\sigma}{|\sigma|}$$

Perfect plasticity

where the plastic (Lagrangian) multiplier λ is a non-negative scalar variable. Combining with Hooke's law expressed, the differential equation for the stress is obtained

$$\dot{\sigma} = E\dot{\epsilon} - \lambda E \frac{\sigma}{|\sigma|}$$

The problem formulation is complemented by the so-called **elastic-plastic loading criteria**. It follows that the general format of the loading criteria is

$$\lambda \geq 0, \quad \Phi(\sigma) \leq 0, \quad \lambda\Phi(\sigma) = 0$$

Elastic-plastic tangent stiffness relation

Considering the plastic state defined by Φ (σ) = 0, $\dot{\Phi}$ > *0* is not admissible, due to the constraint Φ = 0, the plastic multiplier λ is determined from the consistency condition $\dot{\Phi}$ ≤ 0:

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \sigma}\dot{\sigma} \leq 0$$

(rate of changing of the yield criterion)

Inserting the differential equation for the stress ($\dot{\sigma}$) into this inequality leads to

$$\dot{\Phi} = \frac{\sigma}{|\sigma|}E\left(\dot{\epsilon} - \lambda\frac{\sigma}{|\sigma|}\right) = \frac{\sigma}{|\sigma|}E\dot{\epsilon} - E\lambda \leq 0$$

Perfect plasticity

- Plastic loading (*L*) is defined by the situation $\lambda > 0$ and $\dot{\Phi} = 0$, in which case we may solve the previous inequality for λ to obtain

$$\lambda = \frac{\sigma}{|\sigma|}\dot{\epsilon}$$

this is a valid solution only when $(\sigma/|\sigma|)\,\dot{\varepsilon} > 0$, which is the appropriate loading criterion, that must be satisfied in order for plastic strain to evolve.

Perfect plasticity

- Plastic unloading (U) is defined by the situation $\lambda = 0$ and $\dot{\Phi} \leq 0$, obtained when $(\sigma/|\sigma|)\, \dot{\varepsilon} \leq 0$, which is the appropriate loading criterion, that must be satisfied in order for plastic strain to evolve.

As $\dot{\epsilon}^{\mathrm{p}} = \lambda \dfrac{\partial \Phi}{\partial \sigma} = \lambda \dfrac{\sigma}{|\sigma|}$, it follows that the rate equation for the internal variable ε^{p} in terms of the control variable ε is expressed as

$$\dot{\epsilon}^{\mathrm{p}} = \dot{\epsilon} \quad (L), \qquad \dot{\epsilon}^{\mathrm{p}} = 0 \quad (U)$$

Which in turns leads to tangent stiffness relation

$$\dot{\sigma} = 0 \quad (L), \qquad \dot{\sigma} = E\dot{\epsilon} \quad (U)$$

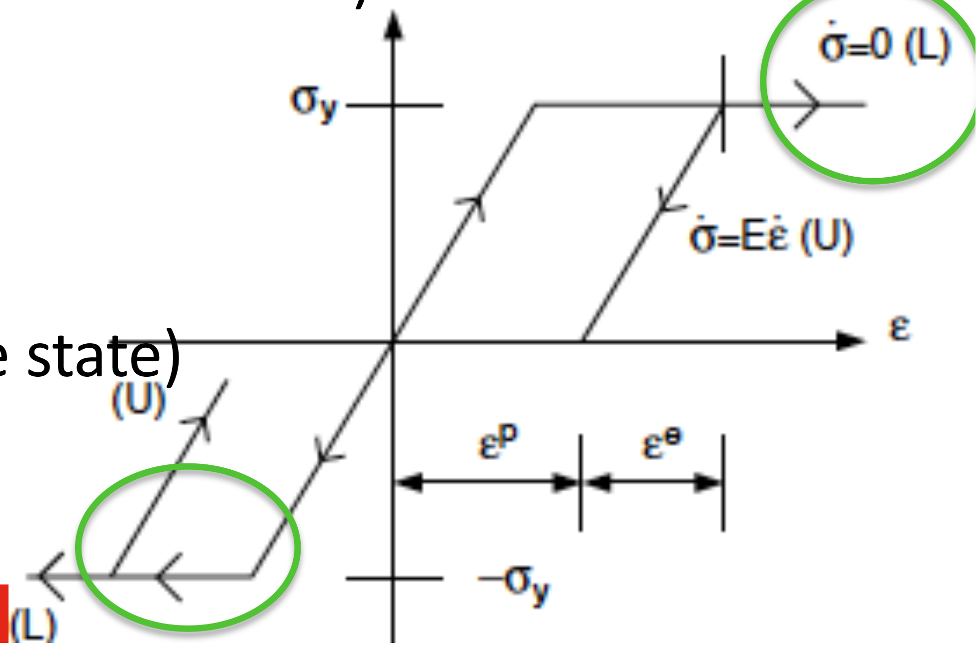
Perfect plasticity

When the yield criterion is satisfied, i.e. when $|\sigma| = \sigma_y$, two different situations are possible:

- The first situation is characterized by $\dot{\varepsilon}$ and σ having the same sign, which gives plastic loading (*L*). The solution is then $\dot{\varepsilon}^p = \dot{\varepsilon}$ and $\dot{\sigma} = 0$, which can be expected for perfectly plastic behavior (as shown in stress-strain curve) for which the tangent stiffness is zero.

- Remark :

the internal work $\sigma_{ij}\varepsilon_{ij} = 0$ (unstable state)



GEOMAS

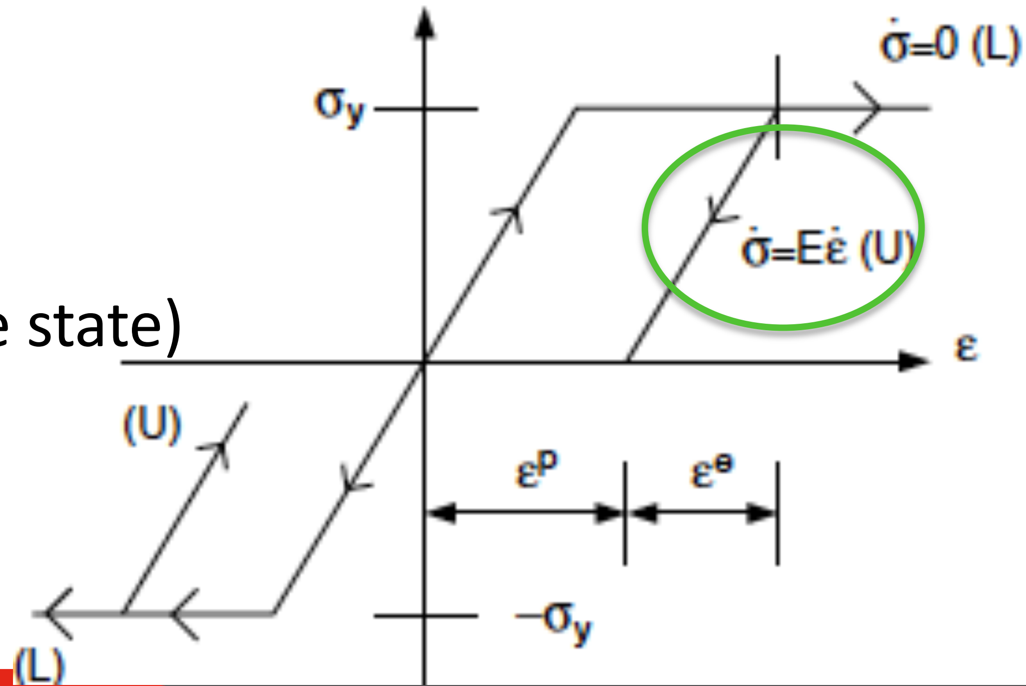
Perfect plasticity

When the yield criterion is satisfied, i.e. when $|\sigma| = \sigma_y$, two different situations are possible:

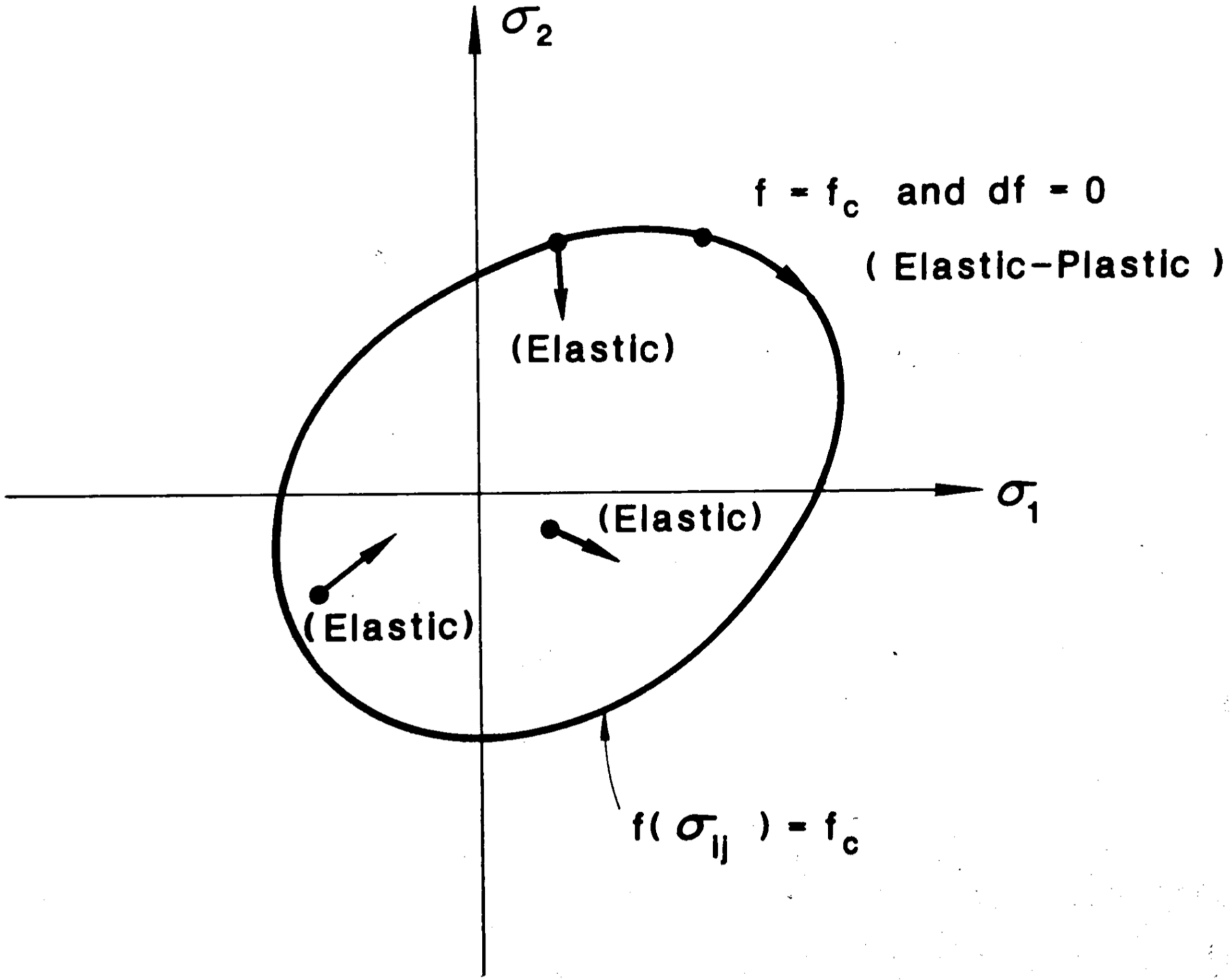
- The second situation is characterized by $\dot{\varepsilon}$ and σ having opposite signs, which gives elastic unloading (U). The solution is then defined by $\dot{\varepsilon}^p = 0$ and $\dot{\sigma} = E\dot{\varepsilon}^e$, which corresponds to elastic response

- Remark

the internal work $\sigma_{ij}\varepsilon_{ij} > 0$ (stable state)



Perfect plasticity



Yield surface for a elastic-perfectly plastic material

Perfect plasticity

- Remark

The expressions introduced for the free energy Ψ and the yield surface Φ are not the only possible ones. For example, we may introduce two internal variables (ε^p and k) and set

$$\Psi = \frac{1}{2}E(\epsilon - \epsilon^{\mathrm{p}})^2 + \sigma_{\mathrm{y}} k$$

New term

$$\Phi = |\sigma^{\mathrm{p}}| - \kappa$$

σ_y replaced by κ

The stress σ is still defined by $\sigma = \frac{\partial \Psi}{\partial \epsilon} = E(\epsilon - \epsilon^{\mathrm{p}})$

While the conjugated variables σ^p and κ (that are the energy conjugate variables to ε^p and κ) are

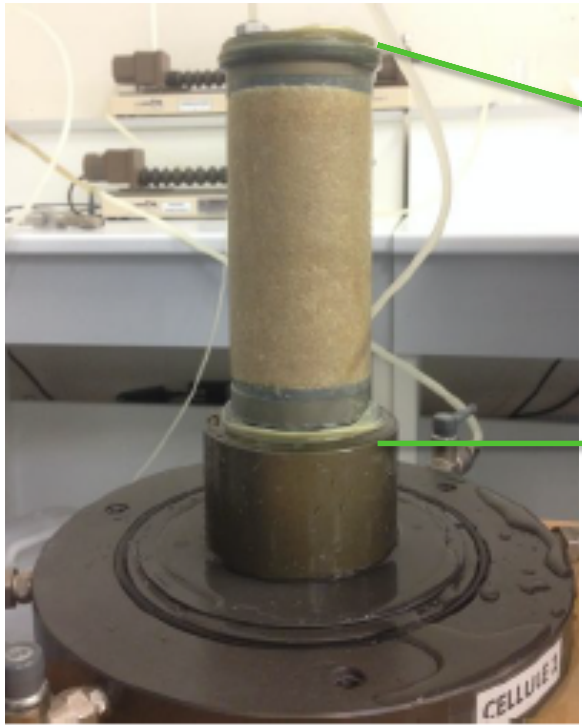
$$\sigma^{\mathrm{p}} = -\frac{\partial \Psi}{\partial \epsilon^{\mathrm{p}}} = E(\epsilon - \epsilon^{\mathrm{p}}) \equiv \sigma \qquad \kappa = -\frac{\partial \Psi}{\partial k} = \sigma_{\mathrm{y}}$$

Same model with more complexity

PLASTICITY IN SOIL MECHANICS

1. Nonlinear plasticity
2. Isotropic hardening
3. Single yield surface plasticity
4. Yield surface / bounding surface
5. Multiple yield surfaces
6. Cyclic behaviour

Sample prep



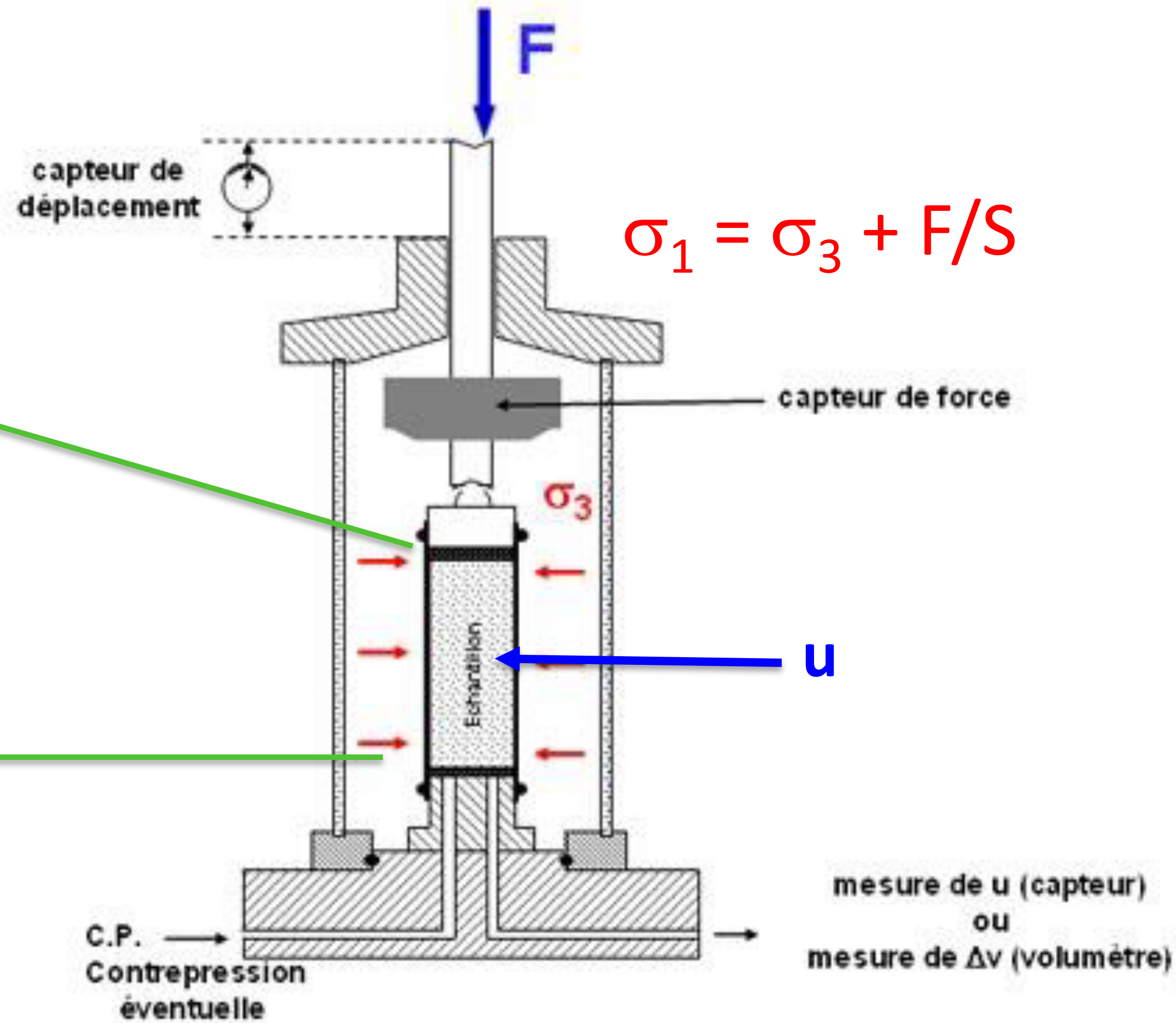


Figure 9.5 : Principe de l'appareil triaxial de révolution

1. Nonlinar plasticity

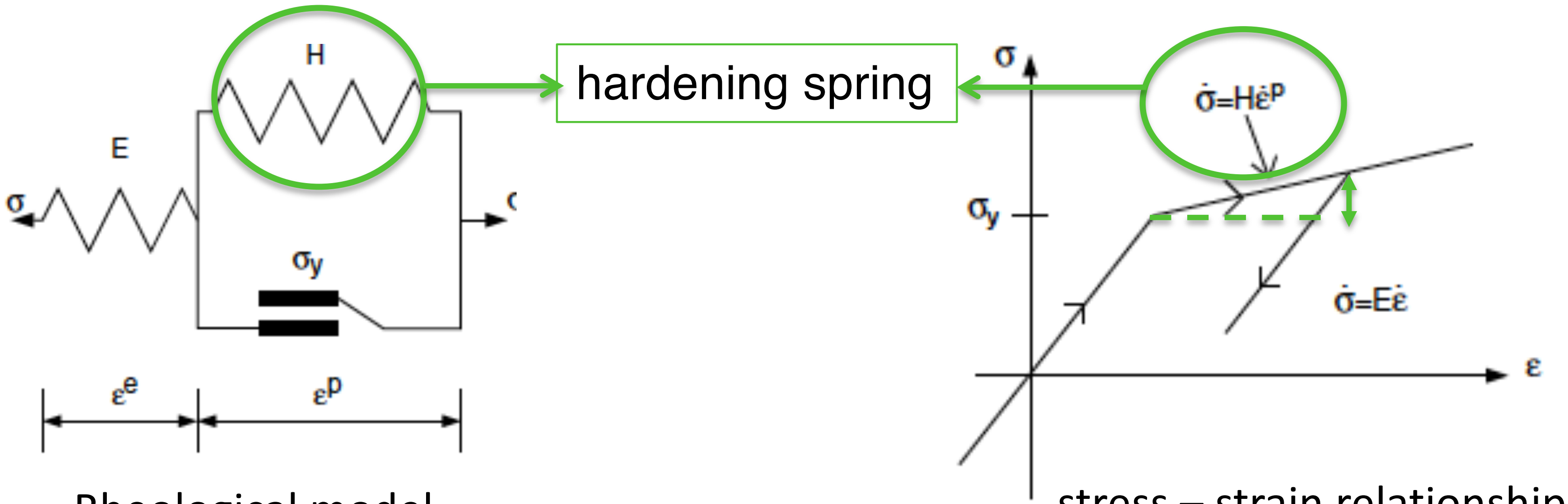
1.1 Free energy and constitutive relations

The free energy per unit volume of a dissipative material is defined as $\Psi(\varepsilon, k_\alpha)$, where ε is the (macroscopic) strain, whereas k_α constitute a finite set of, say N, internal variables that represent irreversible microstructural processes in the material. A typical example (that we shall consider later in more detail) is the plastic deformation that is caused by the relative displacement of grains.

From $\Psi(\varepsilon, k_\alpha)$ we may calculate the stress σ and the socalled dissipative stresses κ_α (that are energy-conjugated to k_α) from the constitutive equations:

$$\sigma = \frac{\partial \Psi}{\partial \epsilon}, \quad \kappa_\alpha \stackrel{\text{def}}{=} -\frac{\partial \Psi}{\partial k_\alpha}, \quad \alpha = 1, 2, \ldots, N$$

Thermodynamic basis — Yield criterion



Rheological model

stress – strain relationship

The frictional-plastic slider is now increasing its resistance due to the amount of slip developed. More specifically, the excess stress over the initial yield stress is due to the "hardening spring" with stiffness *H* that is related to the plastic strain. Upon unloading and reloading, the slider will thus become inactive until the stress has resumed the previous level during loading, i.e. as long as $|\sigma| < \sigma_y + H\,|\varepsilon^{pl}|$, where $H > 0$ is the (constant) hardening modulus.

This behaviour is typical for hardening plasticity.

GEOMAS

2. Isotropic hardening

Apart from ε^p, we now introduce the (isotropic) hardening variable *k*, such that the free energy density is expressed as

$$\Psi(\varepsilon, k) = \frac{1}{2}E(\varepsilon - \varepsilon^p)^2 + \frac{1}{2}\mathrm{H}k^2$$

and the stress σ

$$\sigma = \frac{\partial \Psi(\varepsilon, k)}{\partial \varepsilon} = E(\varepsilon - \varepsilon^p)$$

$$\sigma^p = -\frac{\partial \Psi(\varepsilon, k)}{\partial \varepsilon^p} \equiv \sigma$$

Isotropic hardening

whereas the dissipative stress K, associated with k, is

$$\kappa = -\frac{\partial \Psi(\varepsilon, k)}{\partial k} = -Hk$$

The yield function is now defined as

$$\Phi(\sigma, \kappa) = |\sigma| - \sigma_y - \kappa$$

Isotropic hardening

Plastic flow rule

Inelastic deformation can be produced when $\Phi = 0$. The associative flow and hardening rules are then defined as

$$\dot{\epsilon}^{\mathrm{P}} = \lambda \frac{\partial \Phi}{\partial \sigma} = \lambda \frac{\sigma}{|\sigma|}$$

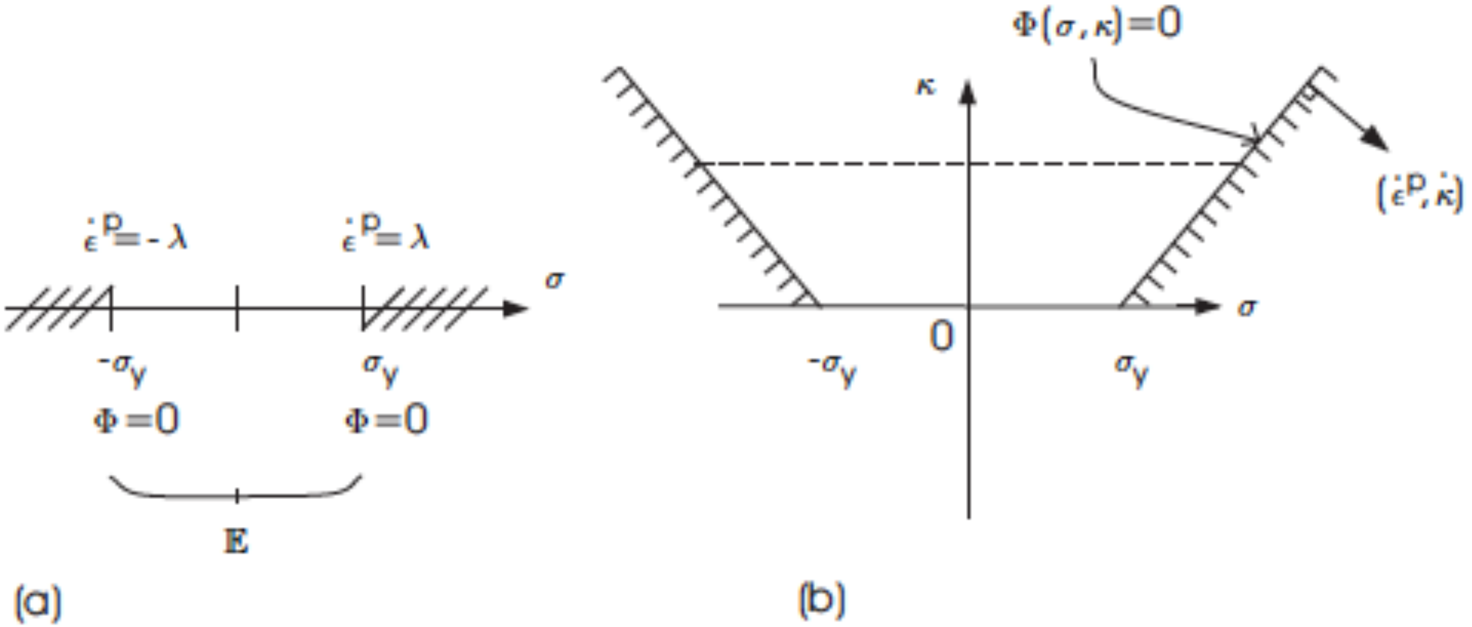
$$\dot{k} = \lambda \frac{\partial \Phi}{\partial \kappa} = -\lambda$$

with

$$\lambda \geq 0, \quad \Phi(\sigma) \leq 0, \quad \lambda \Phi(\sigma) = 0$$

Isotropic hardening

The pair ($\dot{\varepsilon}^p$, $\dot{k}$) can be perceived as the outward pointing normal from the cone defined by $\Phi(\sigma, \kappa) = 0$



(a) Associative flow rule for perfect plasticity, (b) Associative flow and hardening rules for hardening plasticity.

Isotropic hardening

As $\kappa = -\dfrac{\partial \Psi(\varepsilon, k)}{\partial k} = -Hk$ then $\dot{\kappa} = -H\dot{k}$

and $\dot{k} = \lambda \dfrac{\partial \Phi}{\partial \kappa} = -\lambda$

so $\dot{\kappa} = -H\lambda$ (rate of evolution of κ)

Single yield surface plasticity

GEOMAS

Elastoplasticity

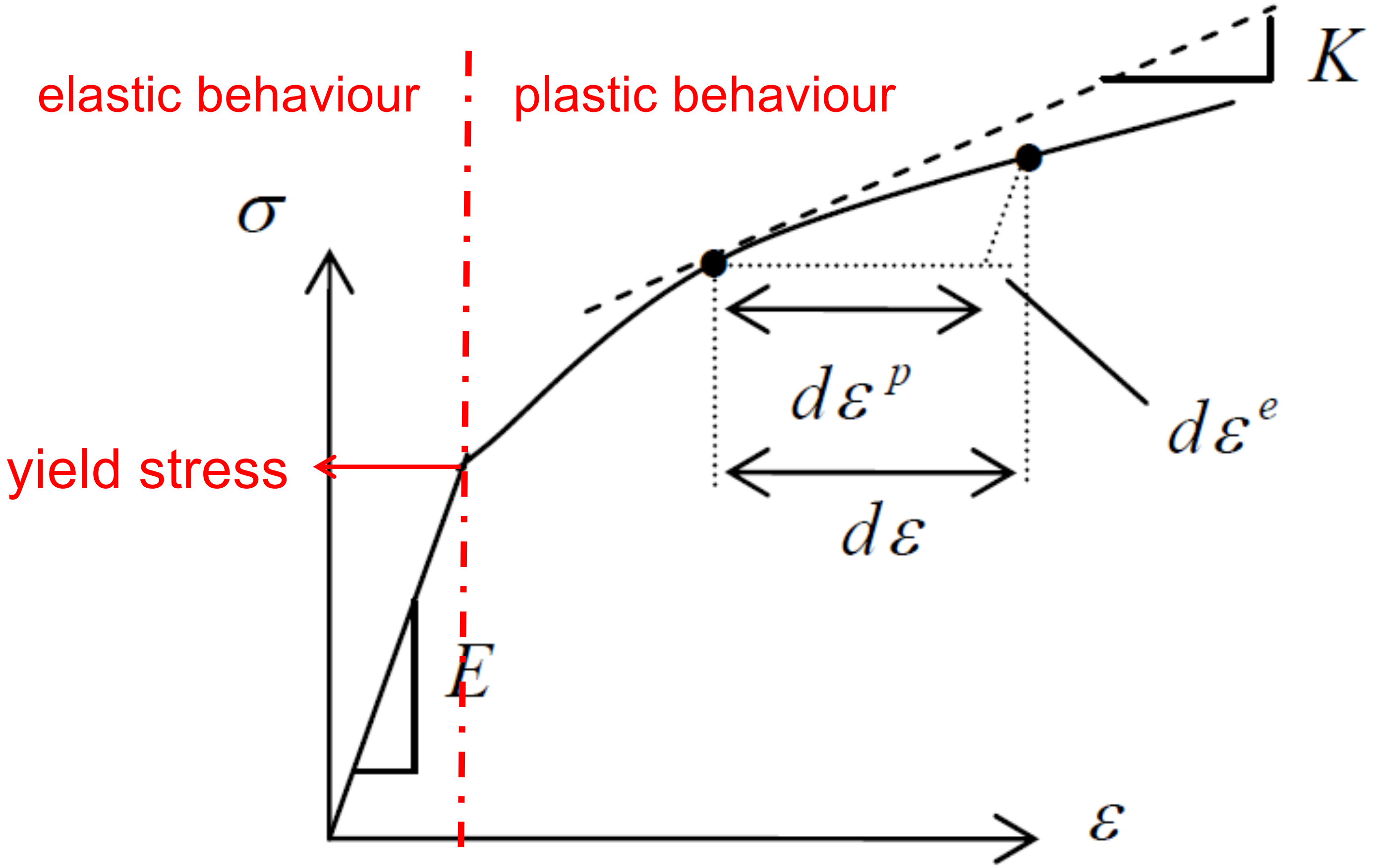
The basic principle of elastoplasticity is that strains are decomposed into an elastic part ε^e and a plastic part ε^p

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

In the stress space the elastic domain is limited by a surface called the yield surface determined by the equation $f(\sigma) = 0$. Inside this domain ($f(\sigma)<0$) the behaviour is purely elastic. When the stress state reaches the limit of this domain and when the stress increment is oriented towards the outside of the domain, plastic strains start to develop

GEOMAS

Non-linear behaviour

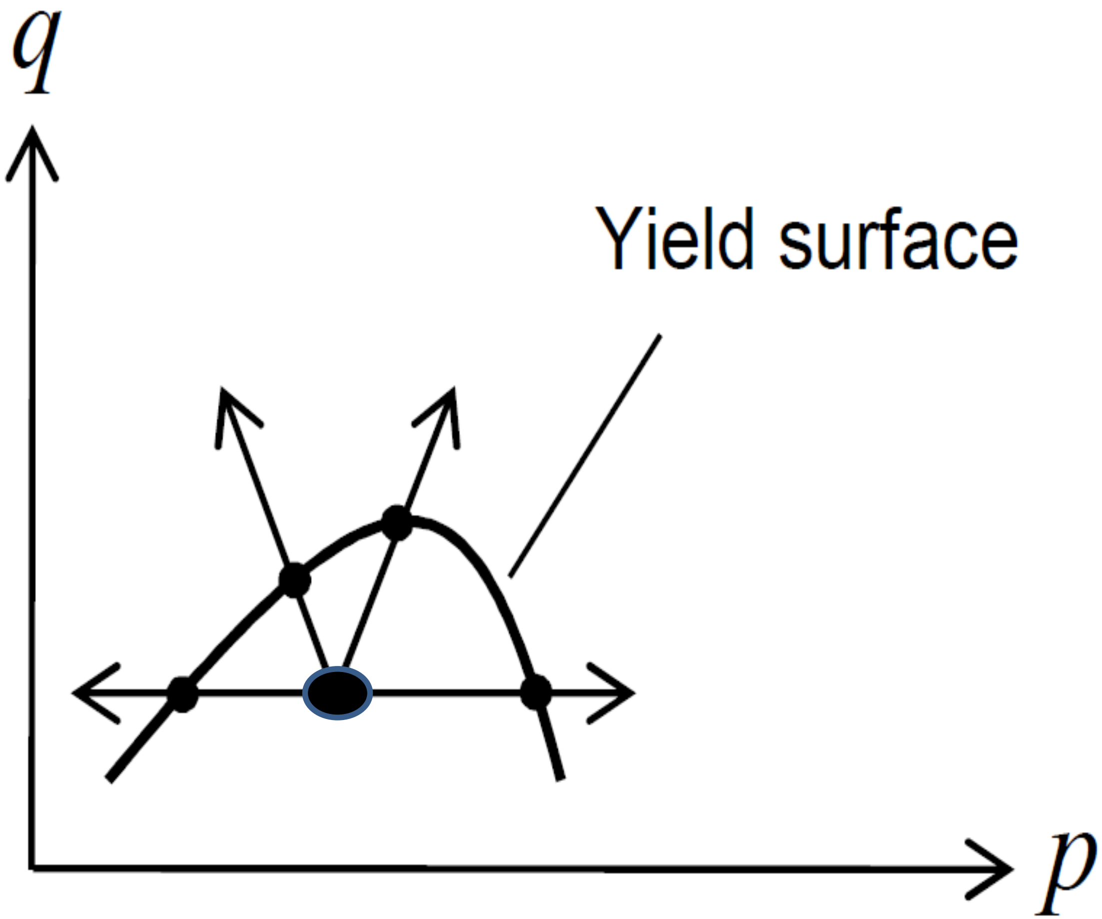


$$d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

Yield surface

Inside the yield surface : elastic behaviour

On the yield surface : plastic behaviour



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Examples of yield surfaces

Pressure-independent behaviour

Von Mises criterion

$$\sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{6}}=k$$

or

$$f(J_2)\equiv J_2-k^2=0$$

Tresca Criterion

$$\max\left\{\frac{1}{2}|\sigma_1-\sigma_2|,\frac{1}{2}|\sigma_2-\sigma_3|,\frac{1}{2}|\sigma_3-\sigma_1|\right\}=k$$

or

$(\sigma_1-\sigma_3) = k'$ with $\sigma_1>\sigma_2>\sigma_3$

Examples of yield surfaces

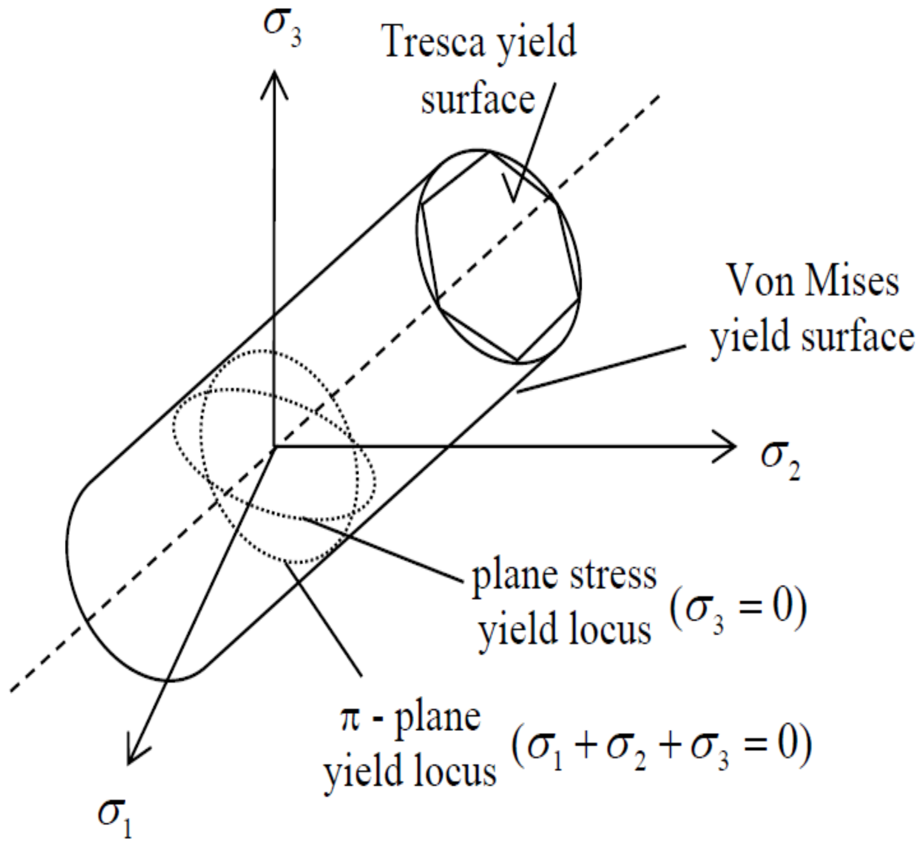


Figure 8.3.12: The Von Mises and Tresca yield surfaces

The Von Mises yield surface is a circular cylinder with axis along the space diagonal
The Tresca yield surface is a similar hexagonal cylinder

Examples of yield surfaces

Pressure-dependent behaviour

Drucker-Prager criterion

$$f(I_1, J_2) \equiv \alpha I_1 + \sqrt{J_2} - k = 0$$

$\alpha = 0$ $\longrightarrow$ Von Mises criterion

Mohr-Coulomb criterion

$$f(\sigma) = (\sigma'_1 - \sigma'_3) - \sin\phi\,(\sigma'_1 + \sigma'_3) - 2c\cos\phi = 0$$

$\phi = 0$ $\longrightarrow$ Tresca criterion

Examples of yield surfaces

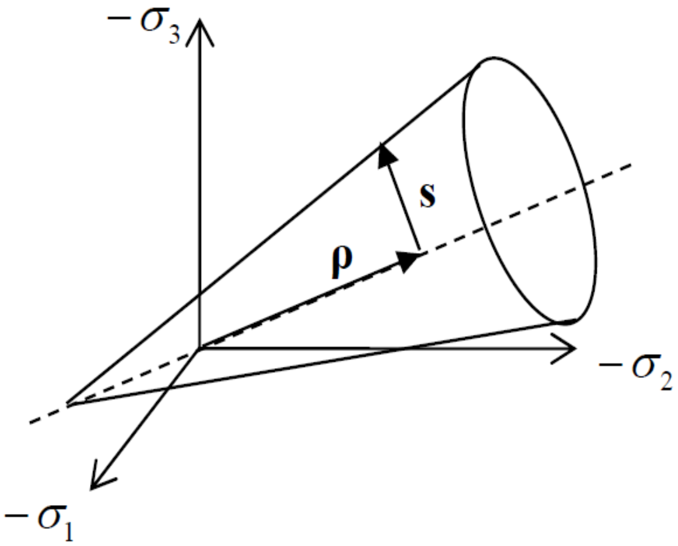


Figure 8.3.15: The Drucker-Prager yield surface

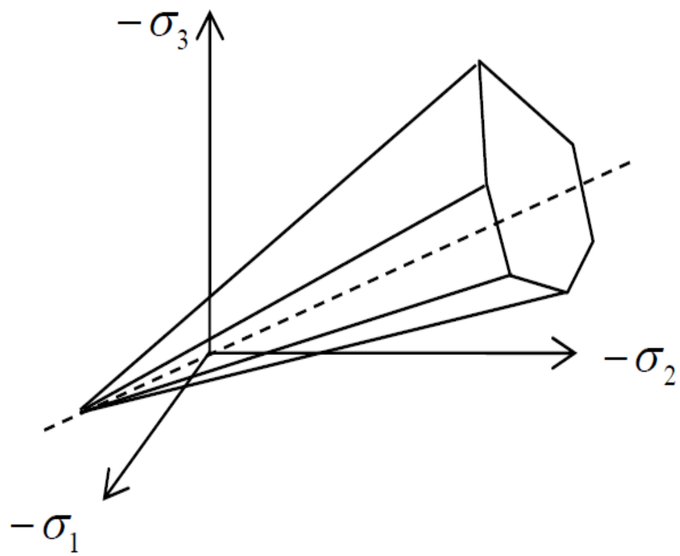
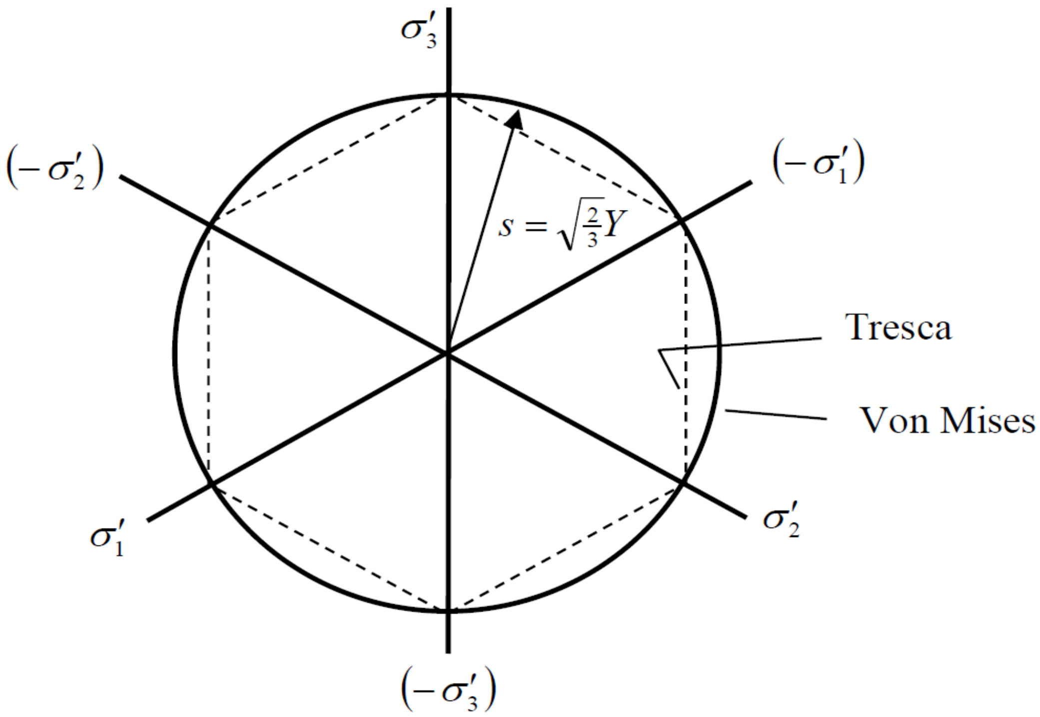
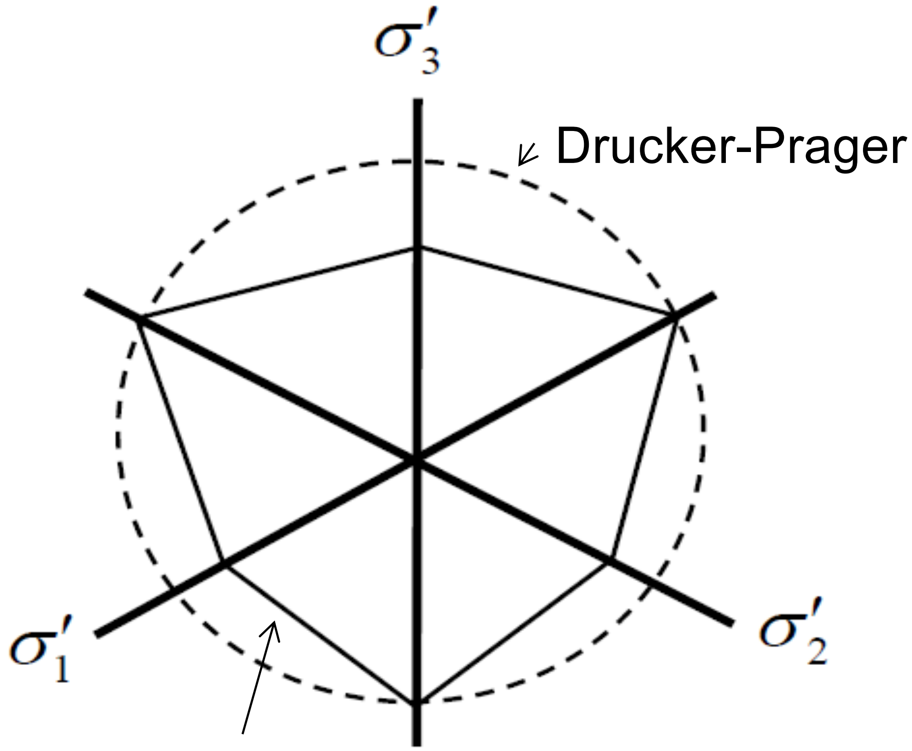


Figure 8.3.20: The Mohr-Coulomb yield surface

Extension of Von Mises and Tresca criteria for pressure-dependent materials

Examples of yield surfaces

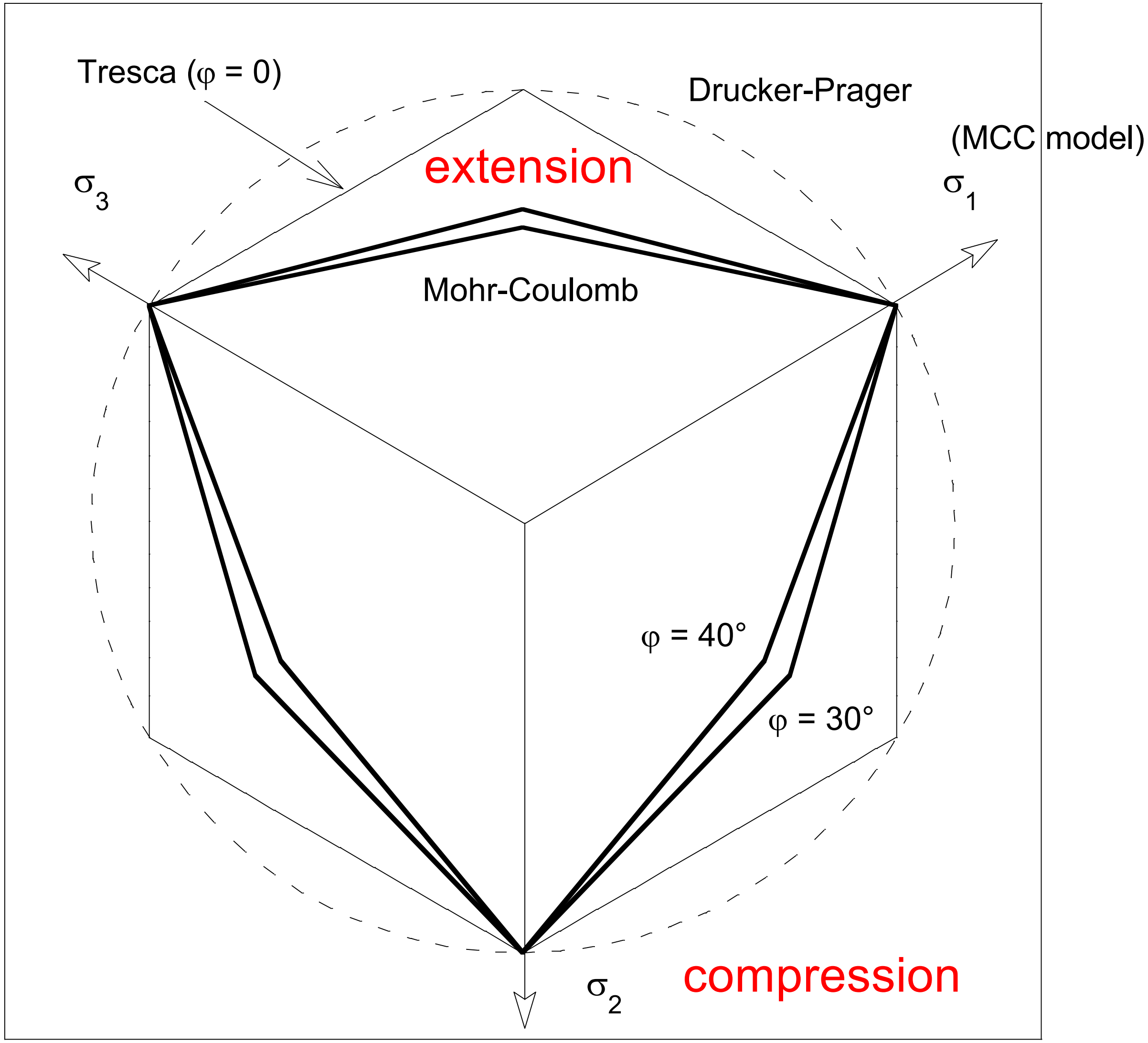




Representation in the octaedral plane

3D Loading

Shape of the failure surface in the octahedral plane



Elastoplasticity

According to the classical theory of plasticity (Hill, 1950), plastic strain rates are proportional to the derivative of the yield function with respect to the stresses. This means that the plastic strain rates can be represented as vectors perpendicular to the yield surface. This classical form of the theory is referred to as associated plasticity.

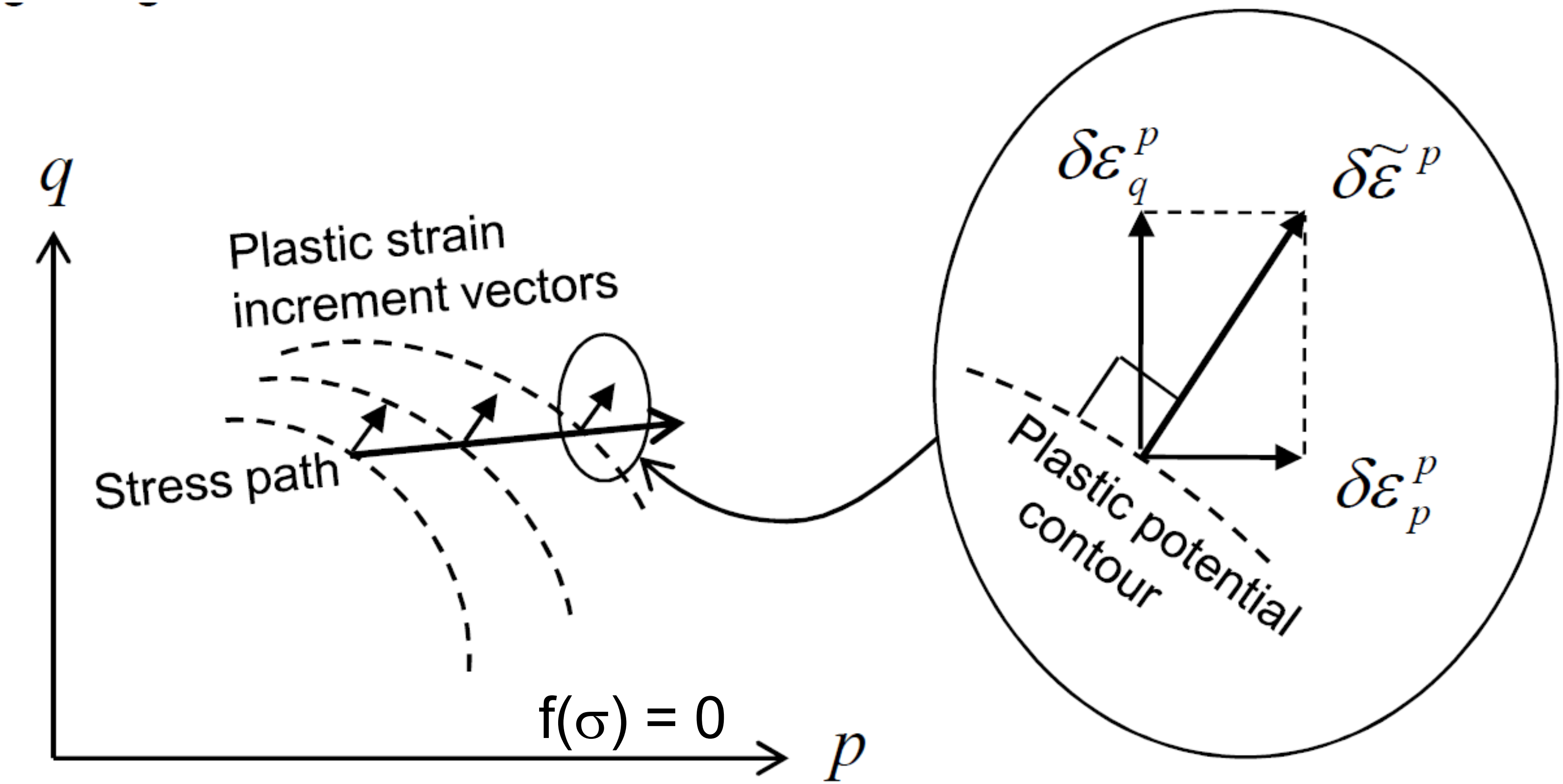
$$d\varepsilon_{ij}^{p} = \lambda\ \delta f/\delta\sigma_{ij}$$

in which λ is the plastic multiplier. For purely elastic behaviour λ is zero, whereas in the case of plastic behaviour λ is positive:

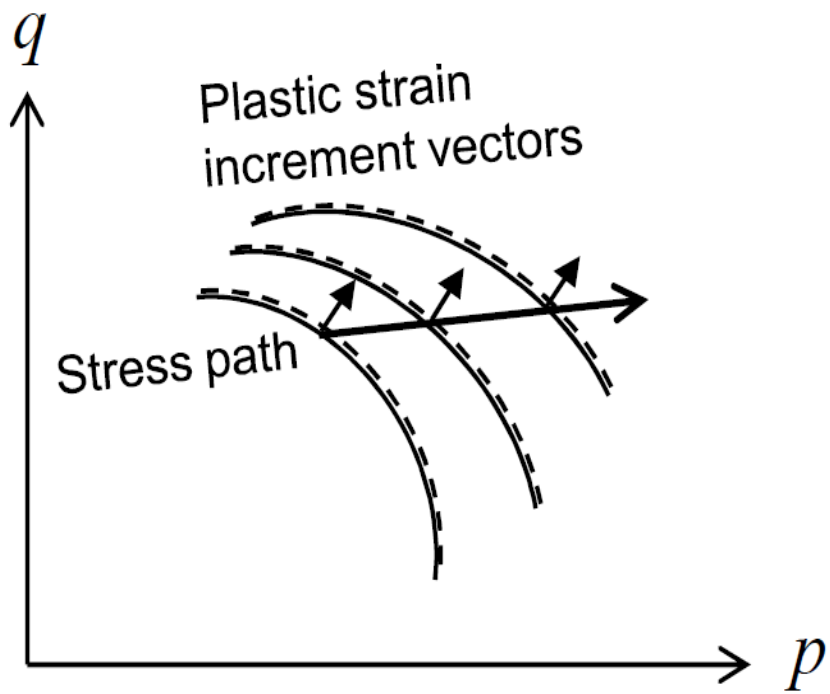
$$\lambda = 0 \quad \text{for:} \quad f < 0 \quad \text{or :} \quad \frac{\partial f}{\partial \underline{\sigma}'}^{T} \underline{\underline{D}}^{e}\, \underline{\dot{\varepsilon}} \leq 0 \quad \text{(Elasticity)} \quad (3.4a)$$

$$\lambda > 0 \quad \text{for:} \quad f = 0 \quad \text{and:} \quad \frac{\partial f}{\partial \underline{\sigma}'}^{T} \underline{\underline{D}}^{e}\, \underline{\dot{\varepsilon}} > 0 \quad \text{(Plasticity)} \quad (3.4b)$$

Yield surface and plastic strain increments



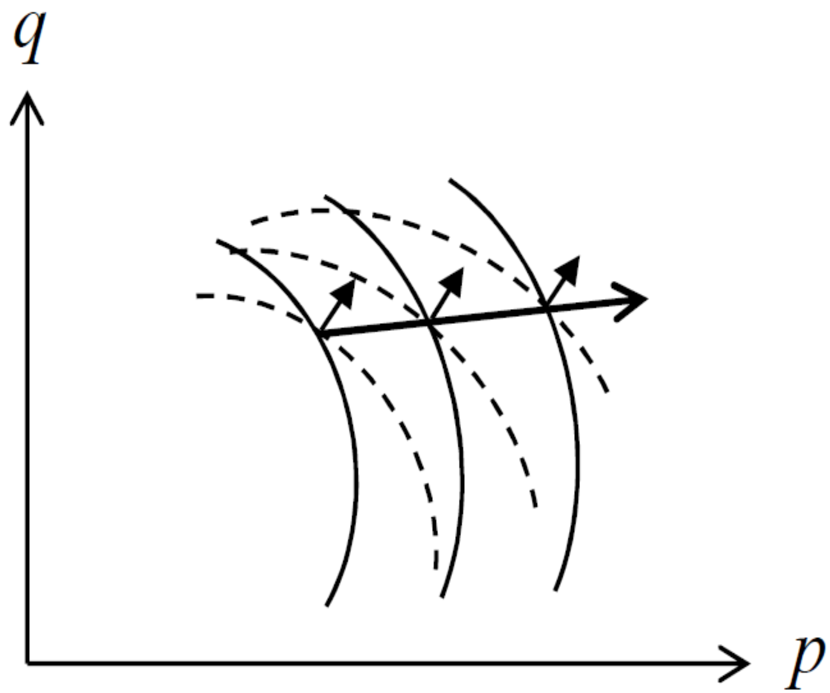
Plastic potential



Associated flow rule

Yield surface
= plastic potential contour

$d\varepsilon^p = \lambda df/d\sigma$

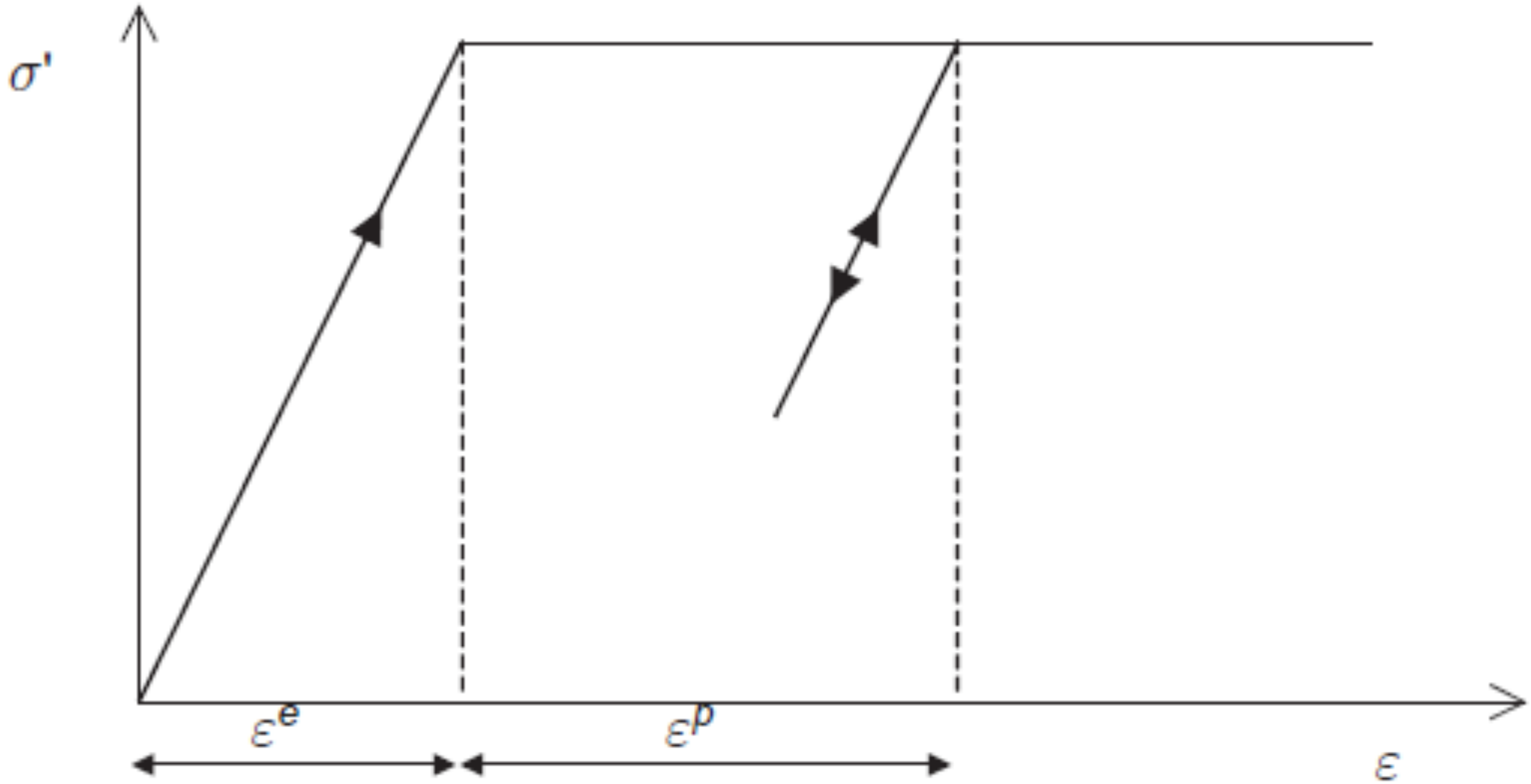


Non-associated flow rule

Yield surface
≠ plastic potential
contour

$d\varepsilon^p = \lambda dg/d\sigma$

Elastic perfectly plastic model



GEOMAS

The Mohr-Coulomb Model

$$f(\sigma) = (\sigma'_1 - \sigma'_3) - \sin\phi\,(\sigma'_1 + \sigma'_3) - 2c\cos\phi = 0$$

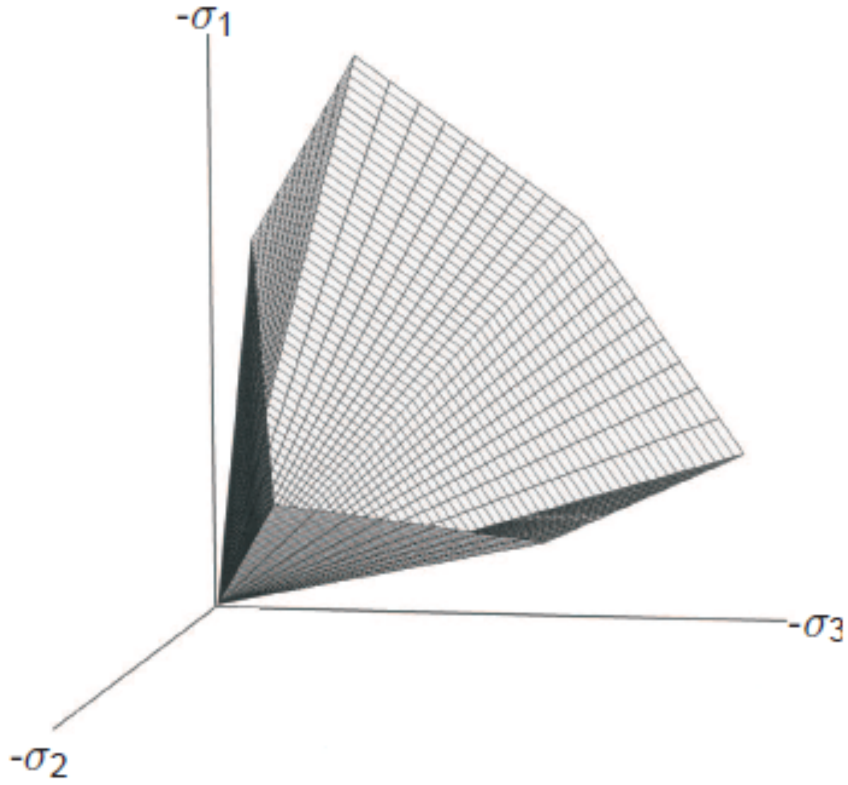


Figure 3.2 The Mohr-Coulomb yield surface in principal stress space (c = 0)

GEOMAS

The Mohr-Coulomb Model

$$f(\sigma) = (\sigma'_1 - \sigma'_3) - \sin\phi\,(\sigma'_1 + \sigma'_3) - 2c\cos\phi = 0$$

Associated flow rule

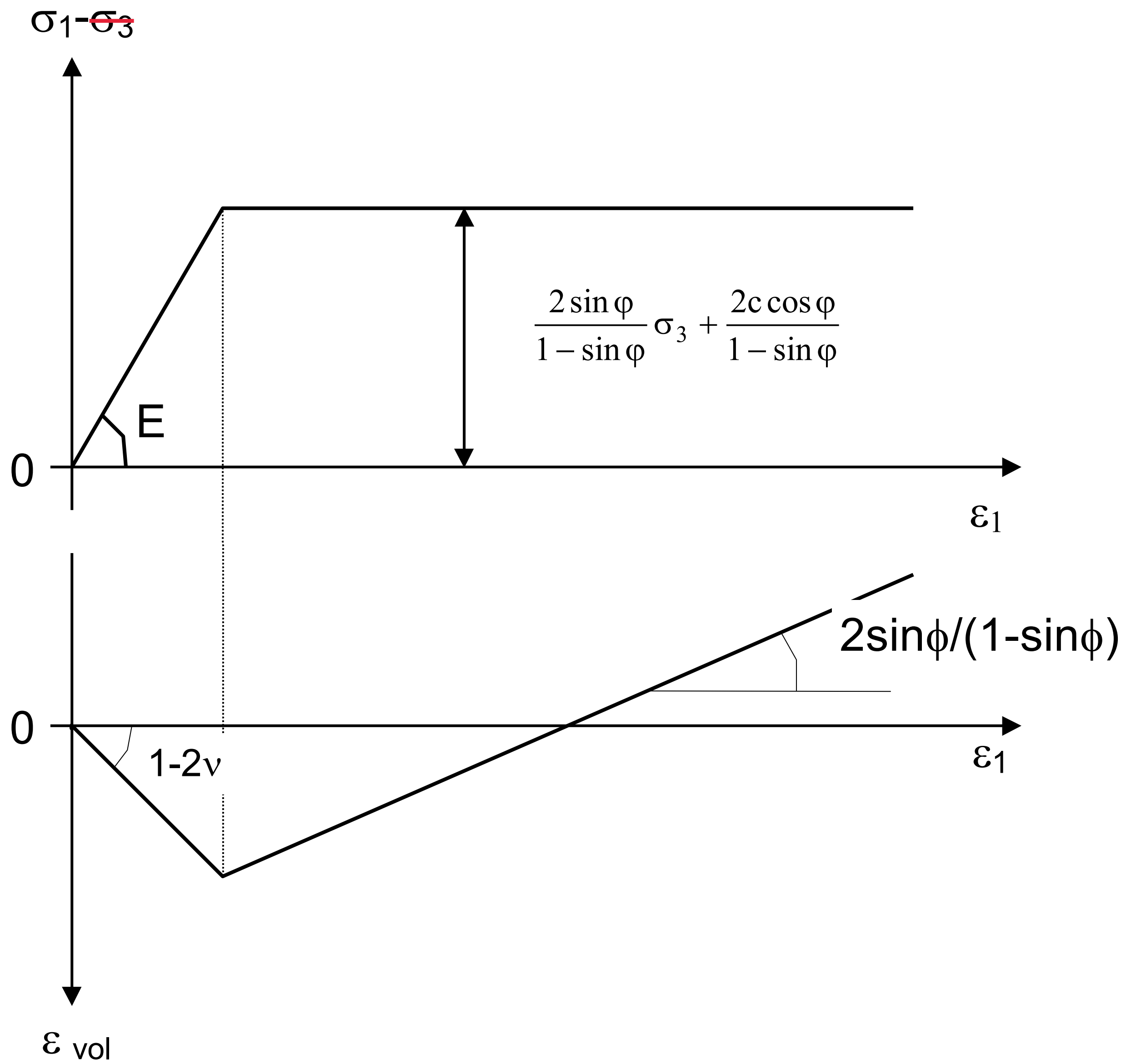
$d\varepsilon^p = \lambda df/d\sigma$

$d\varepsilon^p_1 = \lambda df/d\sigma_1 = \lambda(1- \sin\phi)$

$d\varepsilon^p_3 = \lambda df/d\sigma_3 = -\lambda(1 + \sin\phi)$

$d\varepsilon^p_v = d\varepsilon^p_1 + d\varepsilon^p_3 = -2\lambda\sin\phi$ → dilative behaviour

The Mohr-Coulomb Model



GEOMAS

The Mohr-Coulomb Model

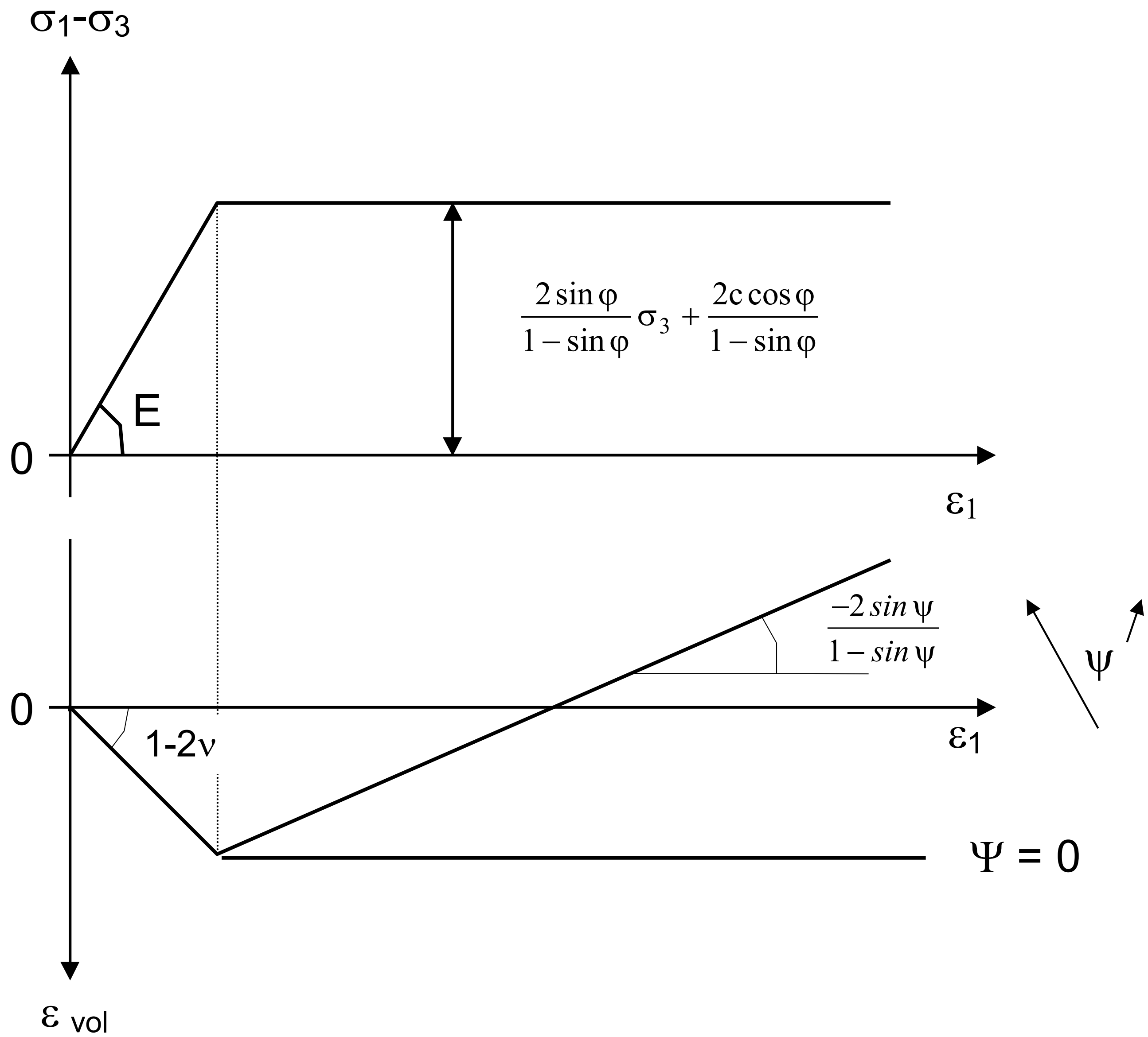
Non-associated flow rule

However, for Mohr-Coulom type yield function, the theory of associated plasticity overestimates dilatancy. Therefore, in addition to the yield function, a plastic potential function *g* is introduced. The case where $g \neq f$ is denoted as non-associated plasticity. In general, the plastic strain rates are written as:

$$d\varepsilon^p = \lambda dg/d\sigma$$

$$g\ (\sigma) = (\sigma'_1 - \sigma'_3) - \sin\psi\ (\sigma'_1 + \sigma'_3)$$

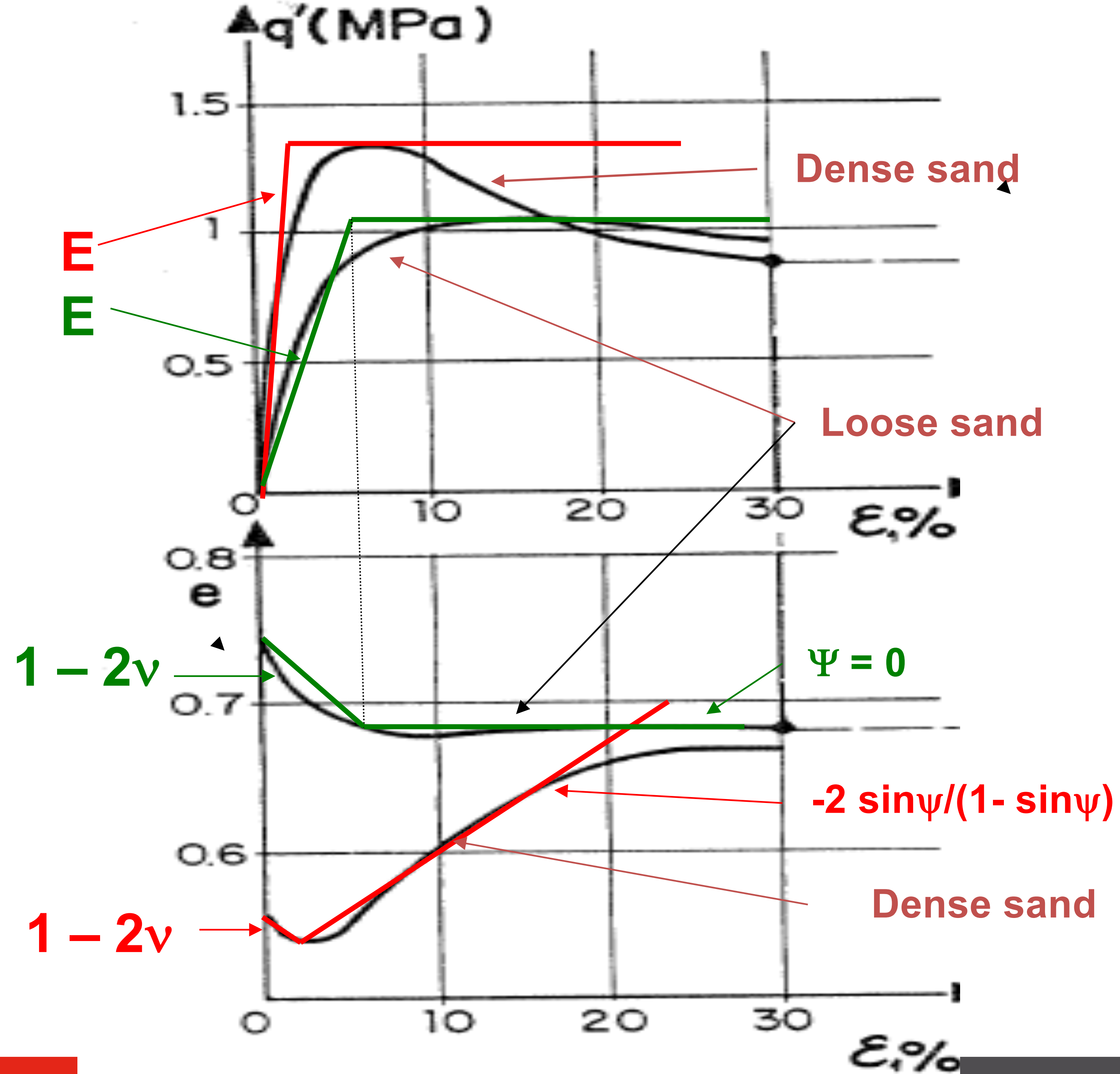
Non-associated Mohr-Coulomb Model



3.3 BASIC PARAMETERS OF THE MOHR-COULOMB MODEL

The linear elastic perfectly-plastic Mohr-Coulomb model requires a total of five parameters, which are generally familiar to most geotechnical engineers and which can be obtained from basic tests on soil samples. These parameters with their standard units are listed below:

E	:	Young's modulus	[kN/m^2]
ν	:	Poisson's ratio	[-]
c	:	Cohesion	[kN/m^2]
φ	:	Friction angle	[°]
ψ	:	Dilatancy angle	[°]



q'(MPa)
1.5
1
0.5
0
10
20
30
ε₁%
E
E
Dense sand
Loose sand
0.8
e
0.7
0.6
1 – 2ν
1 – 2ν
Ψ = 0
-2 sinψ/(1- sinψ)
Dense sand

The Mohr-Coulomb model

Sand	References	γ (kN/m^3)	E (MPa)	ν	c (kPa)	φ °	ψ °)
Hostun (loose)	Mounir (1992)	14	55	0.28	0	35	0.7
Hostun (medium dense)	Mounir (1992)	15.5	85	0.28	0	37	5.5
Hostun (dense)	Mounir (1992)	16.3	95	0.33	0	41	11
Fontainebleau	Ghorbanbeigi (1995)	15.5	40	0.33	0	39	14
Labenne	Mestat *et al*. (1999)	16	33.6	0.28	1	36.5	11.4
Karlsruhe	Arafati (1996)	16	30–45	0.25	0–3	41.6	11.6

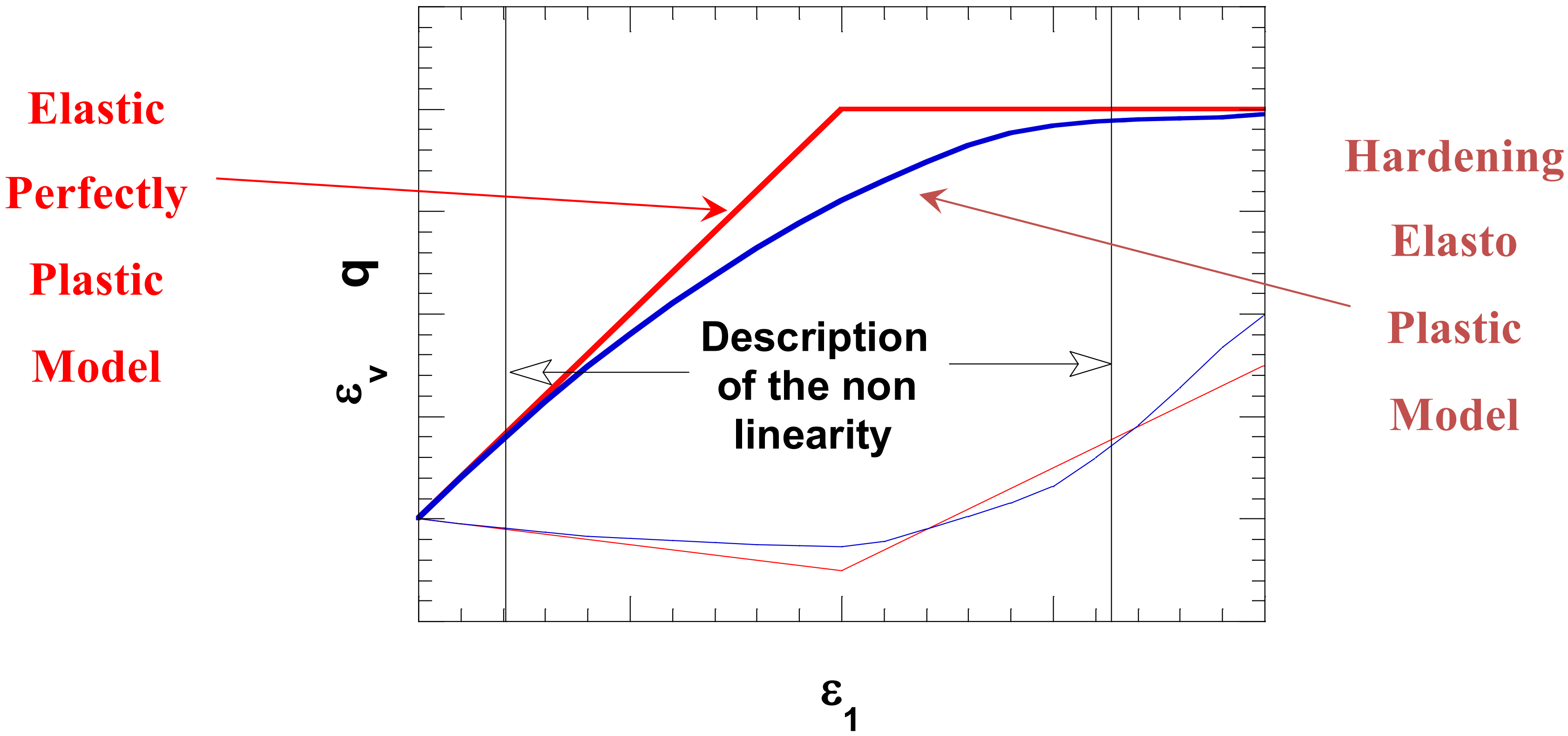
Table 3.7. *Values of the Mohr-Coulomb parameters (sands)*

« IMPROVED » MOHR-COULOMB MODEL

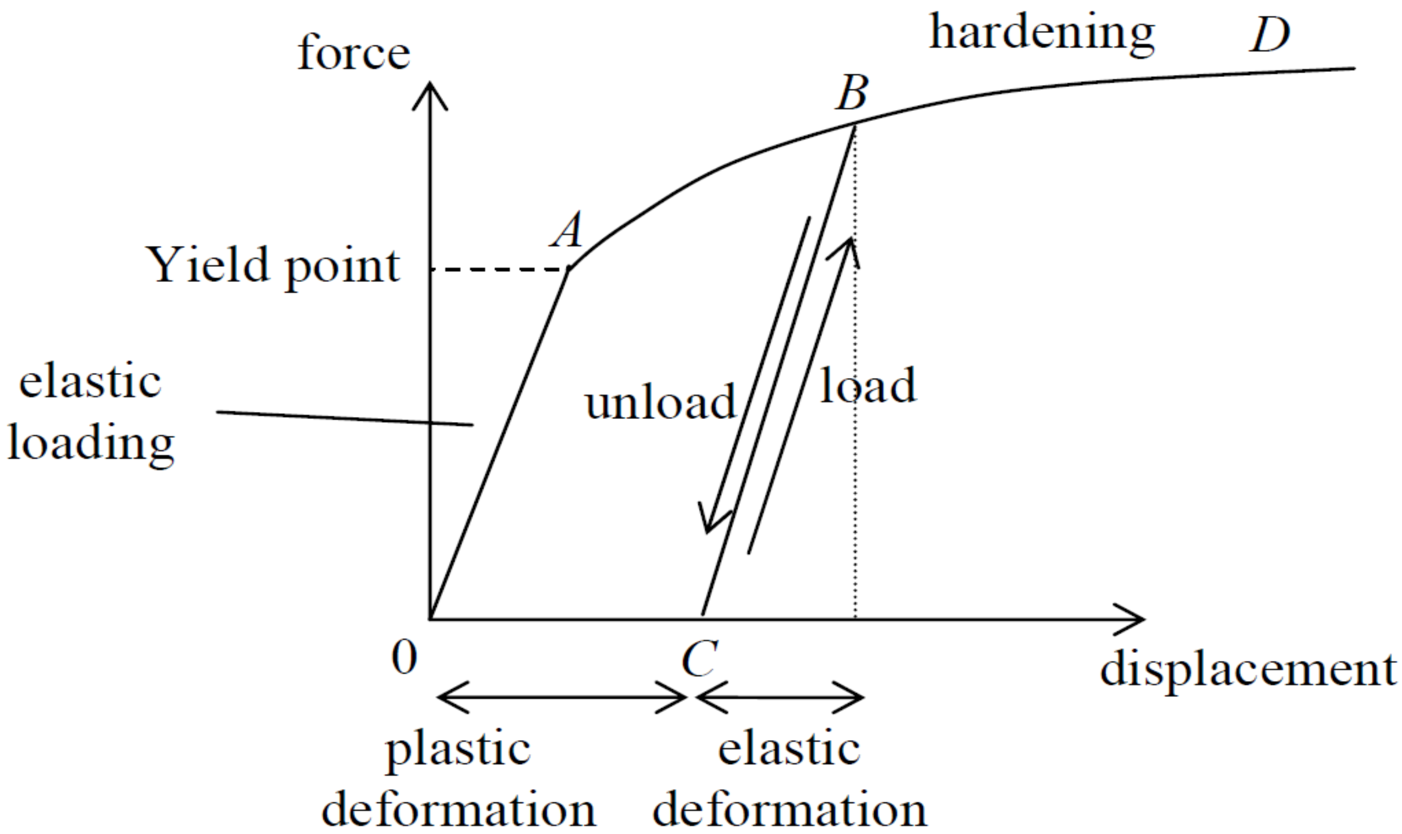
⇓

Description of the non linearity

Elastoplasticity Theory



Elastoplasticity

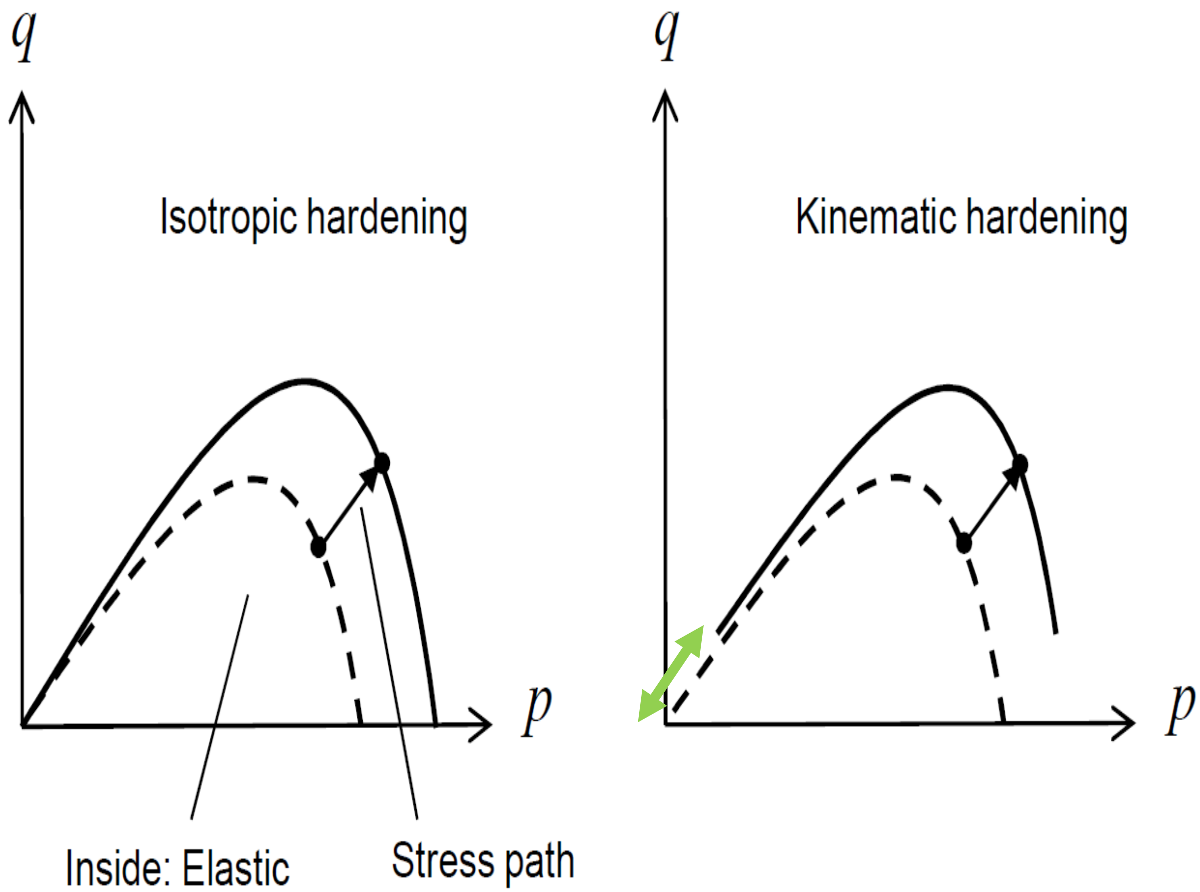


Hardening behaviour

GEOMAS

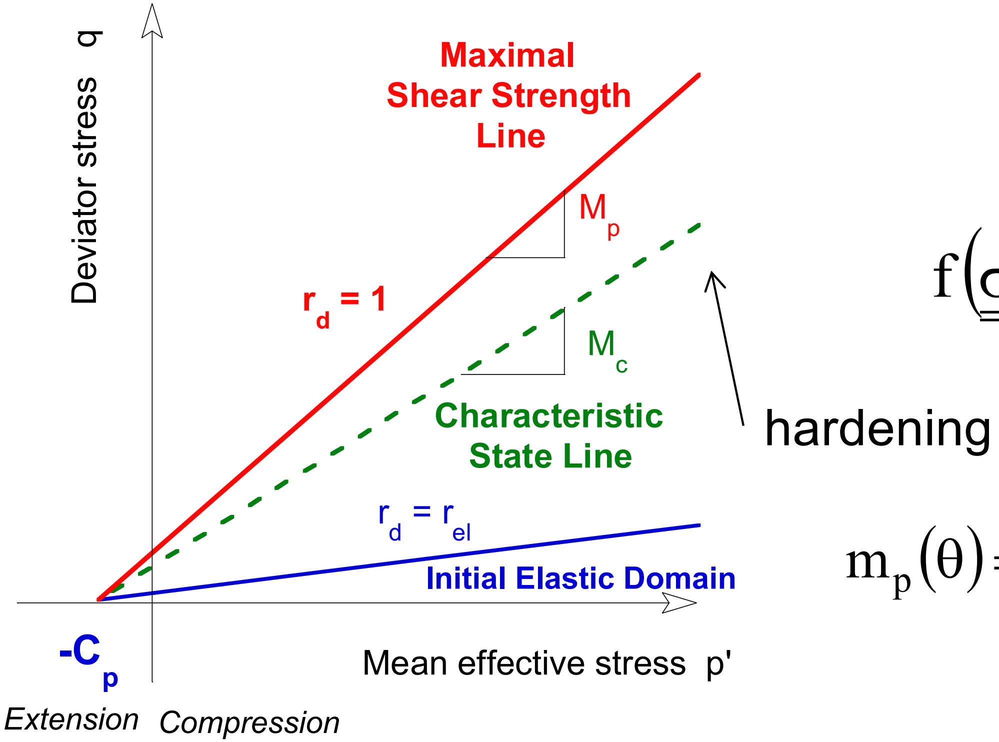
Hardening behaviour : $f(\sigma,\alpha) = 0$

α is the hardening variable



Different types of hardening law

A hardening Mohr-Coulomb model



YIELD SURFACE

$$f(\underline{\underline{\sigma}}, r_d) = \frac{m_p(\theta)}{M_p} \times q - (p' + C_p) \times r_d(\varepsilon_d^p)$$

$$m_p(\theta) = \frac{6}{\sqrt{3}(3 - \sin\varphi')} \times \left[\cos\theta - \frac{\sin\varphi'}{\sqrt{3}} \times \sin\theta\right]$$

(Bardet 1990) θ = Lode 's angle

$$\varphi' = \text{Arc}\sin\left[\frac{3 \times M_p}{6 + M_p}\right]$$

$$c' = \frac{3 - \sin\varphi'}{6 \times \cos\varphi'} \times M_p \times C_p$$

HARDENING FUNCTION

$$r_d(\varepsilon_d^p) = r_{el} + \frac{\varepsilon_d^p \times (1 - r_{el})}{a_{ve} + \varepsilon_d^p}$$

Characteristic state

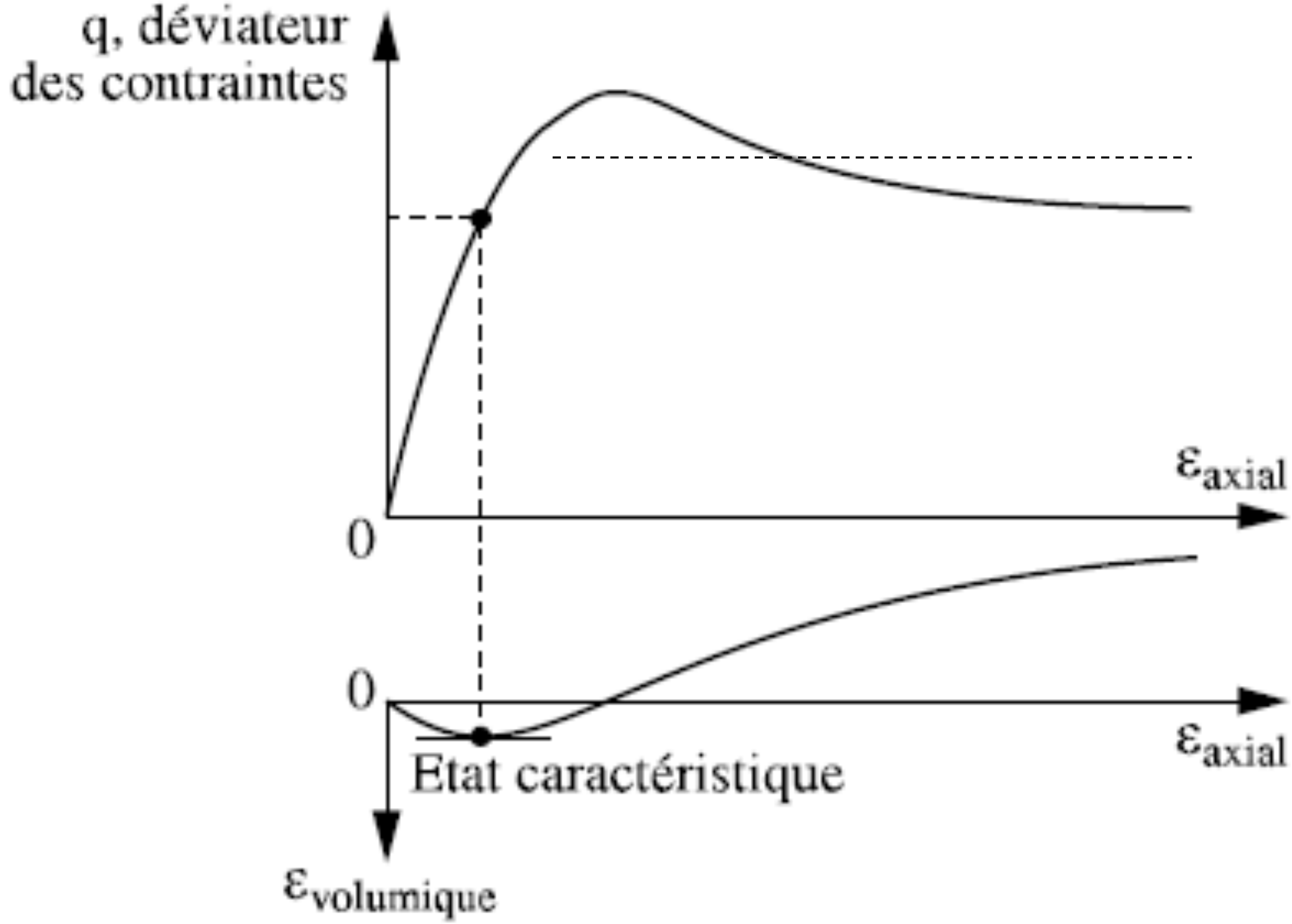


Figure 3.11. *Definition of the characteristic state*

A hardening Mohr-Coulomb model

Non-associated flow rule

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = \frac{M_c}{m_c(\theta)} - \left[\frac{q}{p' + C_p \times \frac{1 - r_d\left(\varepsilon_d^p\right)}{1 - r_{el}}} \right]$$

$$m_c(\theta) = \frac{6}{\sqrt{3}\left(3 - \sin\varphi_c'\right)} \times \left[\cos\theta - \frac{\sin\varphi_c'}{\sqrt{3}} \times \sin\theta \right]$$

$d\varepsilon_v{}^p > 0$ below the characteristic state line : contractive behaviour

$d\varepsilon_v{}^p < 0$ above the characteristic state line : dilative behaviour

A hardening Mohr-Coulomb model

IDENTIFICATION OF THE 7 PARAMETERS

⇓

LABORATORY TRIAXIAL TESTS

E : Tangent modulus — **Curve $q - \varepsilon_1$ effect of p '**

ν : Poisson 's ratio — **Curve $\varepsilon_v - \varepsilon_1$**

M_p : Maximal shear line slope — **(p ', q) diagram**

M_c : Characteristic line slope — **(p ', q) diagram**

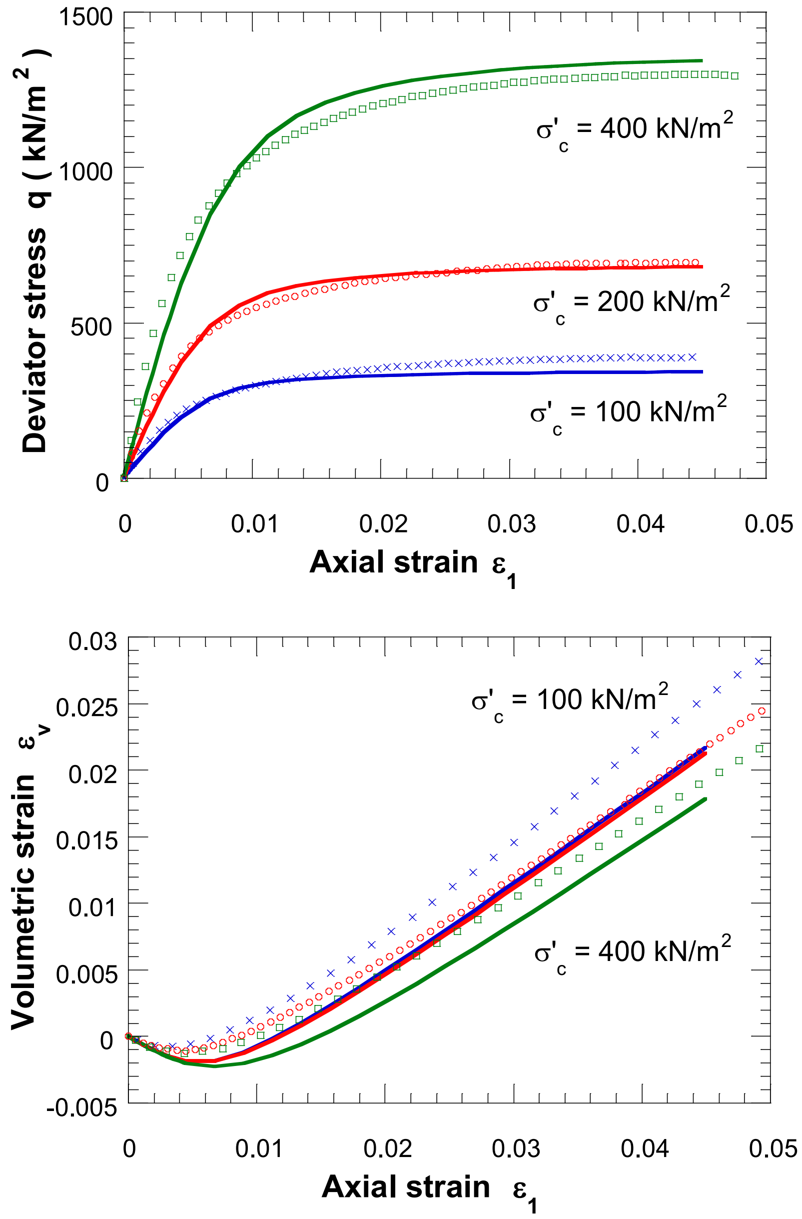
$M_c \approx M_p$ for grouted sands

C_p : Cohesion — **(p ', q) diagram**

r_{el} : Size of elastic domain

a_{ve} : hardening rate

A hardening Mohr-Coulomb model



Uncemented

Fontainebleau Sand

$64 \text{ MPa} \leq E \leq 222 \text{ MPa}$

$\nu = 0.25$

$M_c = 1.17 \Leftrightarrow \varphi_c' = 29.3$ degrees

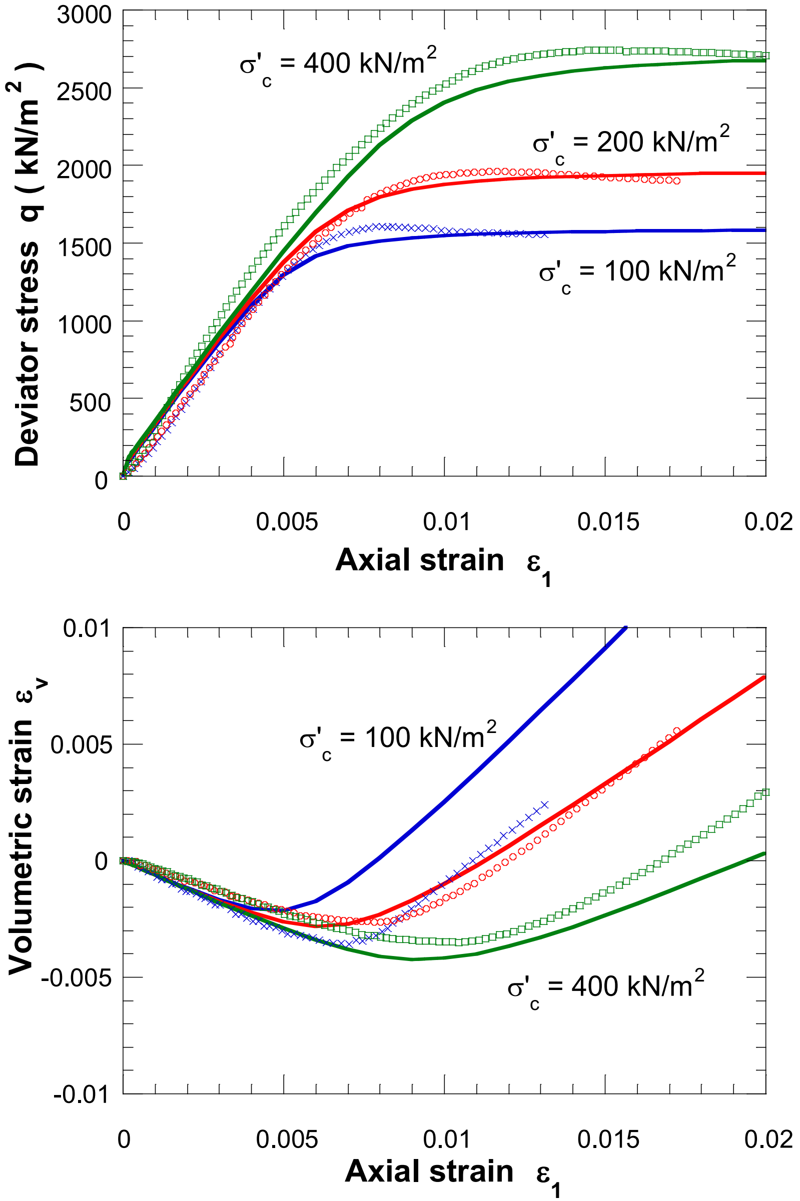
$M_p = 1.60 \Leftrightarrow \varphi' = 39.1$ degrees

$C_p = 0 \Leftrightarrow c' = 0$

$r_{el} = 0.01$

$0.00049 \leq a_{ve} \leq 0.00087$

A hardening Mohr-Coulomb model



Cemented

Fontainebleau Sand

C / W = 0.235

E = 307 MPa

$\nu = 0.20$

$M_c = 1.59 \Leftrightarrow \varphi_c' = 38.9$ degrees

$M_p = 1.68 \Leftrightarrow \varphi' = 41.0$ degrees

$C_p = 317$ kPa $\Leftrightarrow c' = 276$ kPa

$r_{el} = 0.1$

$0.00012 \leq a_{ve} \leq 0.00018$

THE HARDENING SOIL MODEL (ISOTROPIC HARDENING)

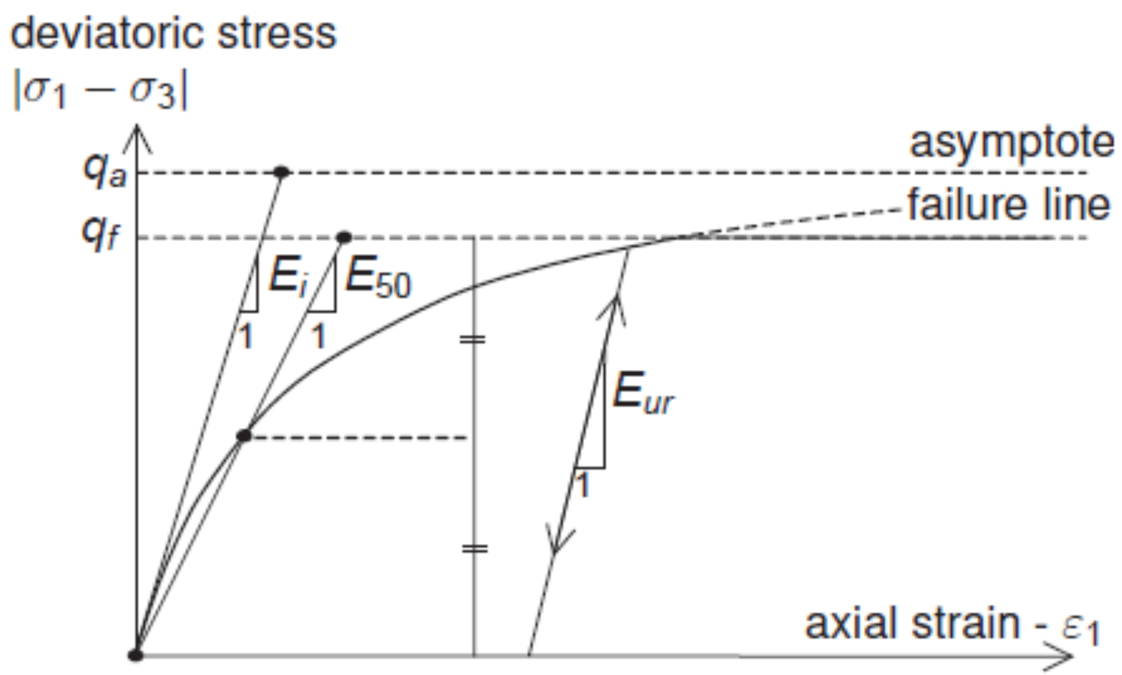
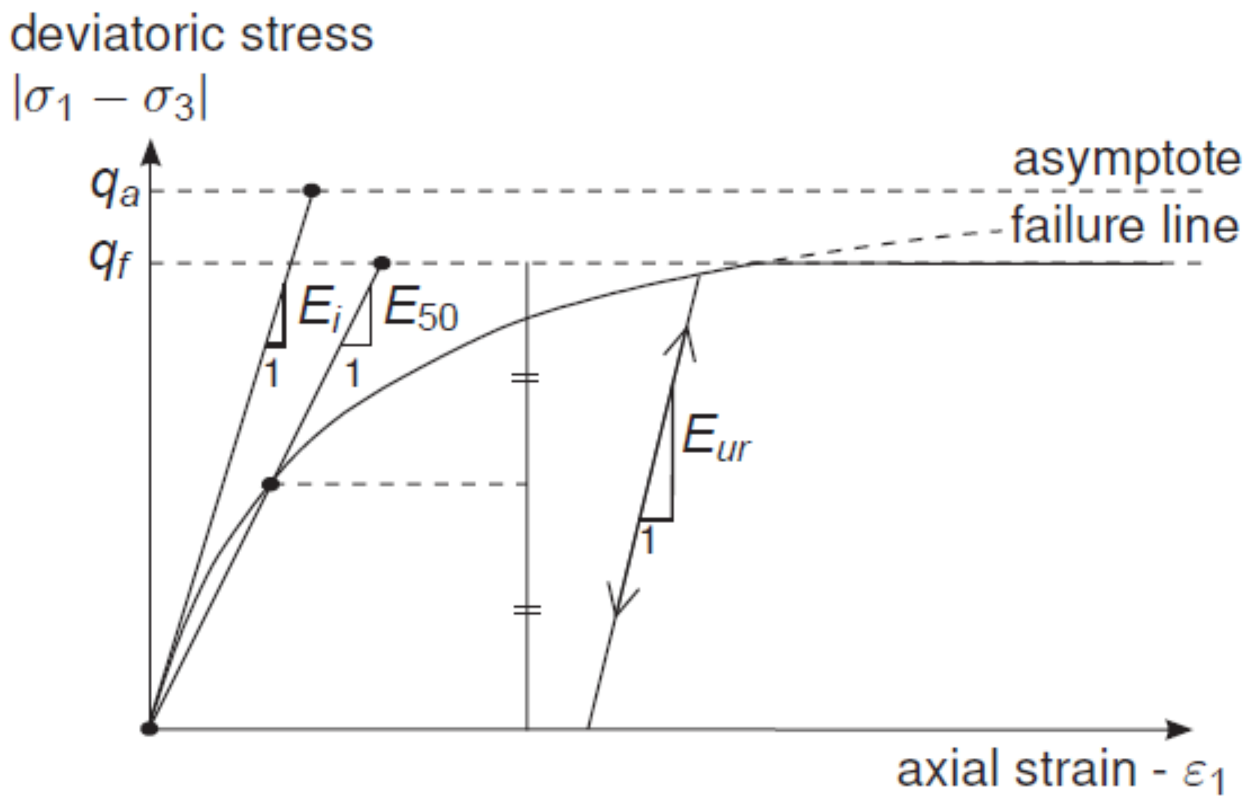


Figure 6.1 Hyperbolic stress-strain relation in primary loading for a standard drained triaxial test

Finite Element Code PLAXIS



Rf

(Plaxis 2D Material Models Manual 2018)

- <u>Equations (triaxial test under drained condition)</u>
  - For $q<q_f$:

$$-\varepsilon_1 = \frac{1}{E_i}\frac{q}{1-q/q_a}$$

  - Value of q_f at failure

$$E_i = \frac{2E_{50}}{2-R_f}$$

$$q_f = (c\cot\varphi - \sigma'_3)\frac{2\sin\varphi}{1-\sin\varphi} \quad \text{and:} \quad q_a = \frac{q_f}{R_f}$$

  - Secant modulus at 50% of the maximun deviatoric stress

$$E_{50} = E_{50}^{ref}\left(\frac{c\cos\varphi - \sigma'_3\sin\varphi}{c\cos\varphi + p^{ref}\sin\varphi}\right)^m$$

  - Unloading / reloading modulus E_{ur}

$$E_{ur} = E_{ur}^{ref}\left(\frac{c\cos\varphi - \sigma'_3\sin\varphi}{c\cos\varphi + p^{ref}\sin\varphi}\right)^m$$

Yield surface

$$f = \bar{f} - \gamma^p$$

where $\bar{f}$ is a function of stress and γ^p is a function of plastic strains:

$$\bar{f} = \frac{2}{E_i}\frac{q}{1 - q/q_a} - \frac{2q}{E_{ur}} \qquad \gamma^p = -(2\varepsilon_1^p - \varepsilon_\nu^p) \approx -2\varepsilon_1^p$$

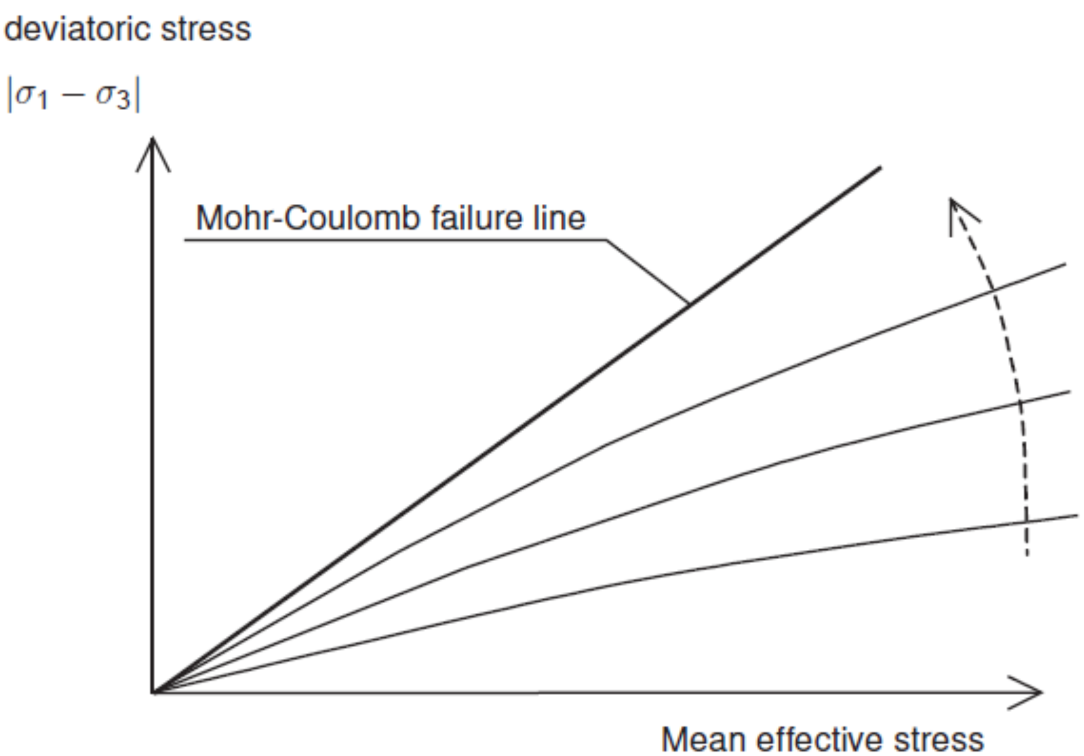


Figure 6.2 Successive yield loci for various constant values of the hardening parameter γ^p

GEOMAS

Potential function

$$\dot{\varepsilon}_v^p = \sin \psi_m \dot{\gamma}^p \tag{6.11}$$

Clearly, further detail is needed by specifying the mobilised dilatancy angle ψ_m. For the present model, the following is considered:

$$\begin{array}{ll} \text{For } \sin \varphi_m < 3/4 \sin \varphi : & \psi_m = 0 \\ \text{For } \sin \varphi_m \geq 3/4 \sin \varphi \text{ and } \psi > 0 & \sin \psi_m = \max\left(\dfrac{\sin \varphi_m - \sin \varphi_{cv}}{1 - \sin \varphi_m \sin \varphi_{cv}}, 0 \right) \\ \text{For } \sin \varphi_m \geq 3/4 \sin \varphi \text{ and } \psi \leq 0 & \psi_m = \psi \\ \text{If } \varphi = 0 & \psi_m = 0 \end{array} \tag{6.12}$$

where φ_{cv} is the critical state friction angle, being a material constant independent of density, and φ_m is the mobilised friction angle:

$$\sin\varphi_m = \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3 - 2c \cot \varphi} \tag{6.13}$$

Failure parameters as in Mohr-Coulomb model (see Section 3.3):

c	:	(Effective) cohesion	[kN/m^2]
φ	:	(Effective) angle of internal friction	[°]
ψ	:	Angle of dilatancy	[°]

Basic parameters for soil stiffness:

E_{50}^{ref}	:	Secant stiffness in standard drained triaxial test	[kN/m^2]
E_{oed}^{ref}	:	Tangent stiffness for primary oedometer loading	[kN/m^2]
E_{ur}^{ref}	:	Unloading / reloading stiffness (default $E_{ur}^{ref} = 3E_{50}^{ref}$)	[kN/m^2]
m	:	Power for stress-level dependency of stiffness	[-]

Drained Triaxial Tests on loose Sand: experiments/HS model simulations

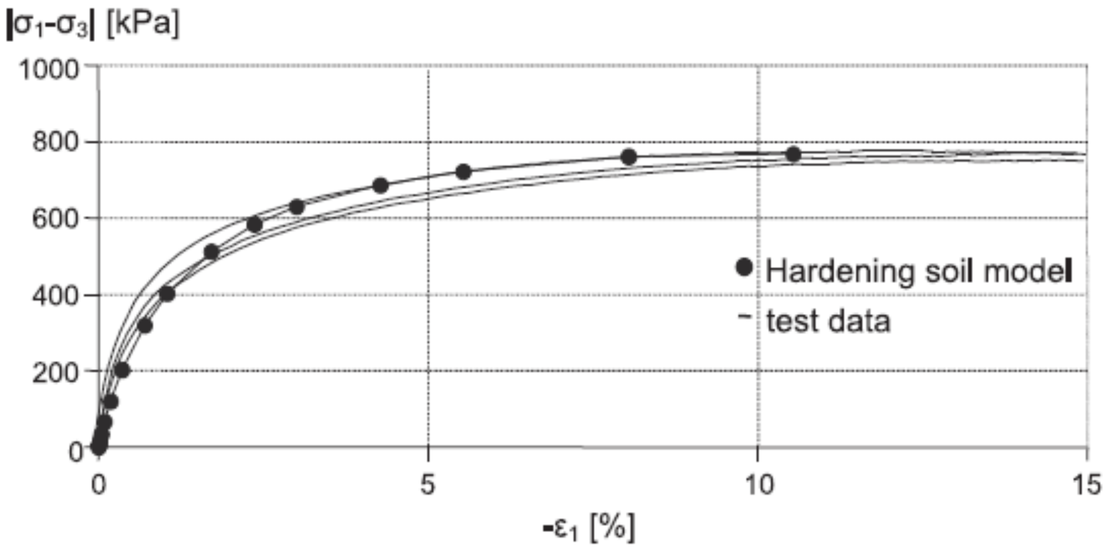


Figure 13.7 Results of drained triaxial tests on loose Hostun sand, principal stress ratio versus axial strain

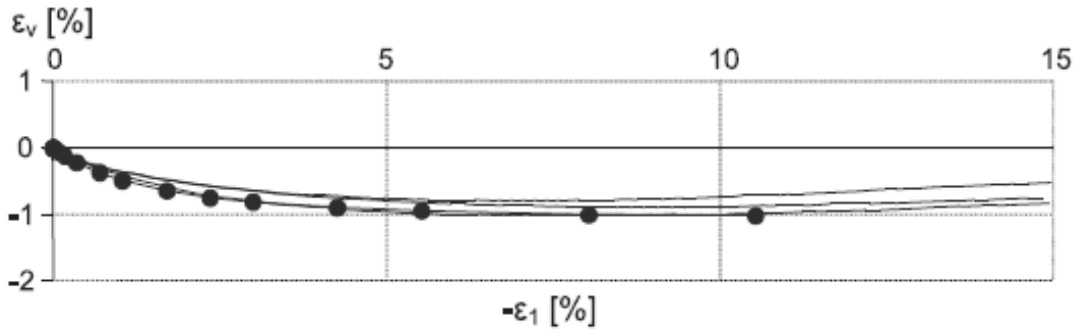


Figure 13.8 Results of drained triaxial tests on loose Hostun sand, volumetric strain versus axial strain

Drained Triaxial Tests on dense Sand: experiments/HS model simulations

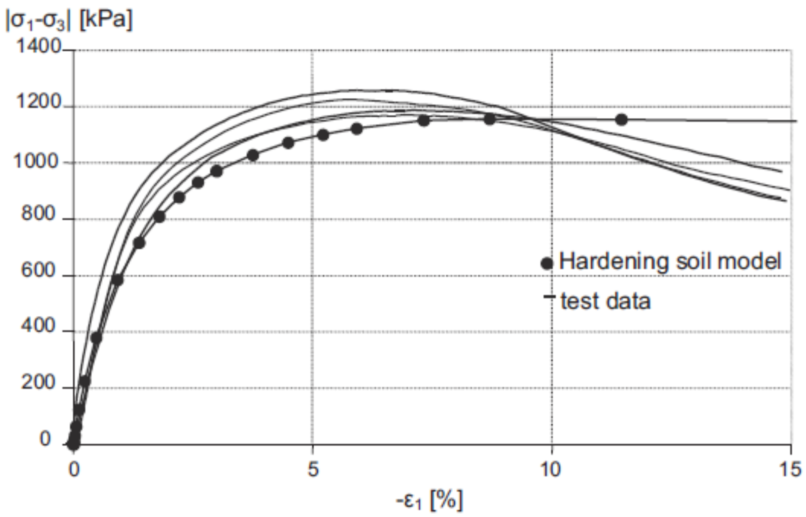


Figure 13.9 Results of drained triaxial tests on dense Hostun sand, principal stress ratio versus axial

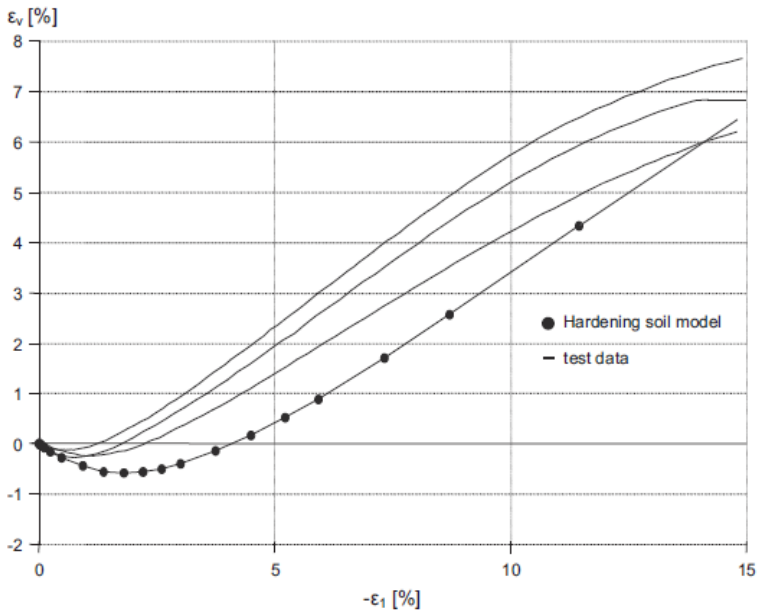
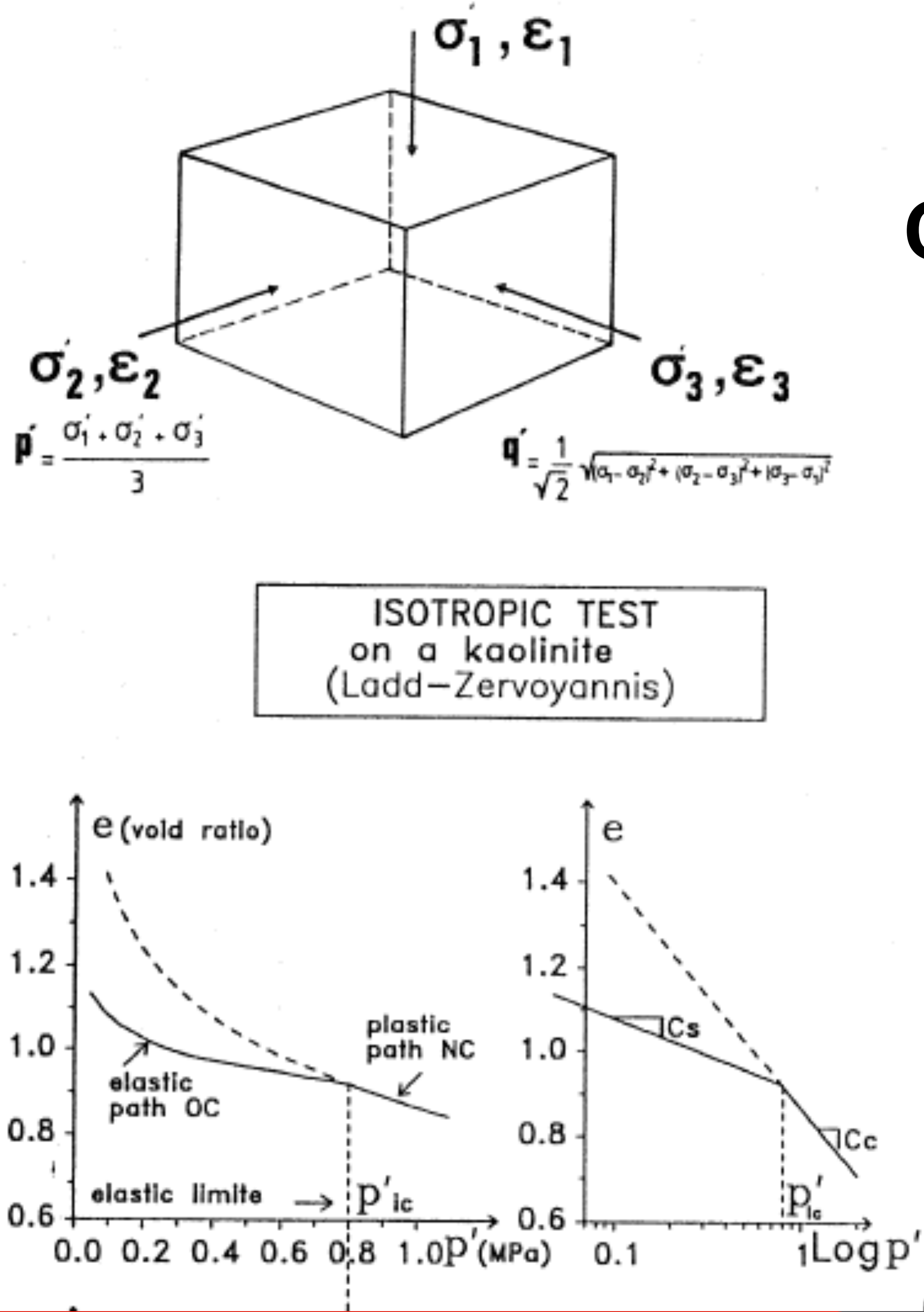


Figure 13.10 Results of drained triaxial tests on dense Hostun sand, volumetric strain versus axial strain



Clay

c_c and c_s in e – log p'

λ and κ in e – ln p'

Modified Cam Clay Model

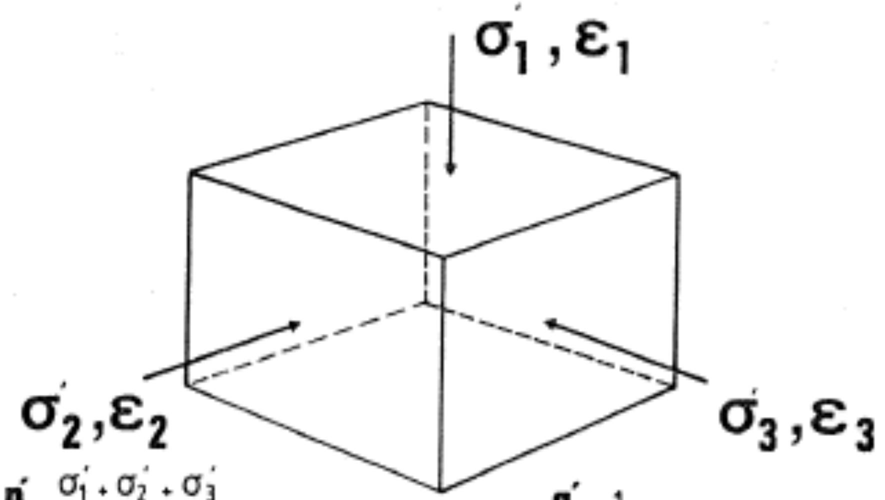
In the Modified Cam-Clay model, a logarithmic relation is assumed between void ratio e and the mean effective stress p' in virgin isotropic compression, which can be formulated as:

$$e - e^0 = -\lambda \ln\left(\frac{p'}{p^0}\right) \quad \text{(virgin isotropic compression)} \qquad (10.1)$$

The parameter λ is the Cam-Clay compression index, which determines the compressibility of the material in primary loading. When plotting relation (Eq. 10.1) in a e - ln p' diagram one obtains a straight line. During unloading and reloading, a different line is followed, which can be formulated as:

$$e - e^0 = -\kappa \ln\left(\frac{p'}{p^0}\right) \quad \text{(isotropic unloading and reloading)} \qquad (10.2)$$

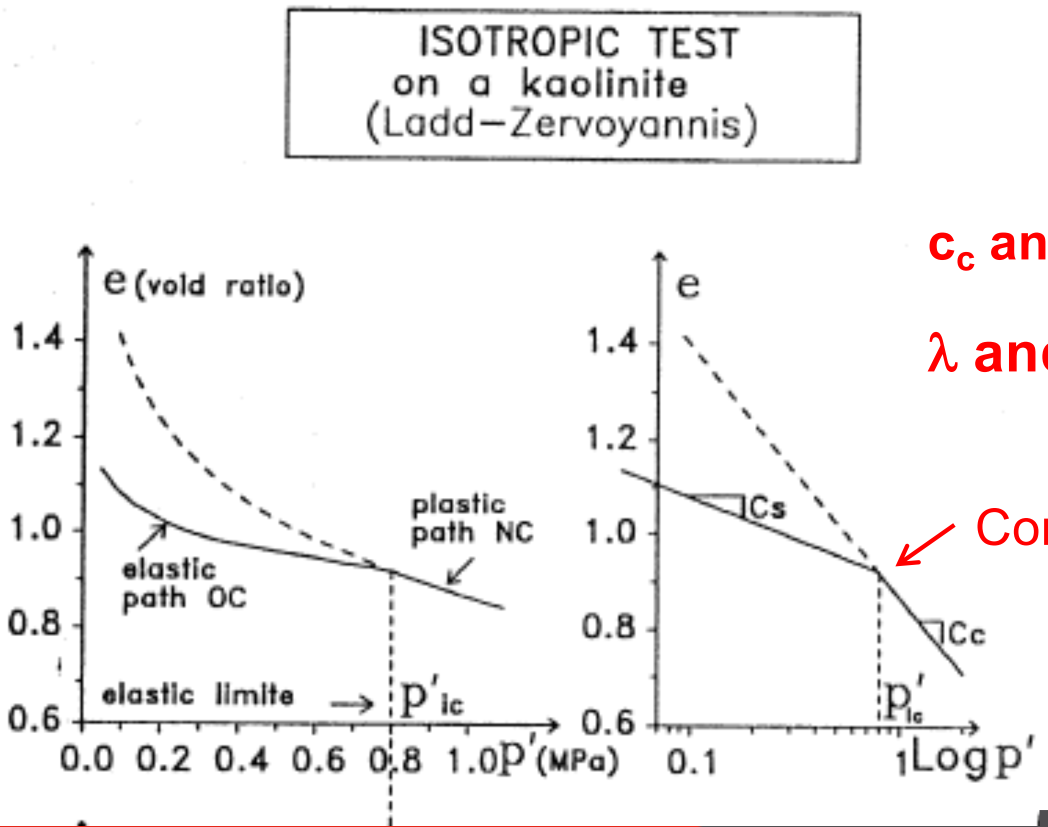
The parameter κ is the Cam-Clay swelling index, which determines the compressibility of material in unloading and reloading. In fact, an infinite number of unloading and reloading lines exists in p' - e-plane each corresponding to a particular value of the preconsolidation stress p_c.



$$p' = \frac{\sigma'_1 + \sigma'_2 + \sigma'_3}{3}$$

$$q' = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Clay



c_c and c_s in e – log p'

λ and κ in e – ln p'

Consolidation stress

The yield function of the Modified Cam-Clay model is defined as:

$$f = \frac{q^2}{M^2} + p'(p' - p_c) \qquad (10.3)$$

p_c is the hardening variable

The yield surface ($f = 0$) represents an ellipse in p' - q-plane as indicated in Figure 10.1. The yield surface is the boundary of the elastic stress states. Stress paths within this boundary only give elastic strain increments, whereas stress paths that tend to cross the boundary generally give both elastic ans plastic strain increments.

In p' - q-plane, the top of the ellipse intersects a line that we can be written as:

$$q = Mp' \qquad (10.4)$$

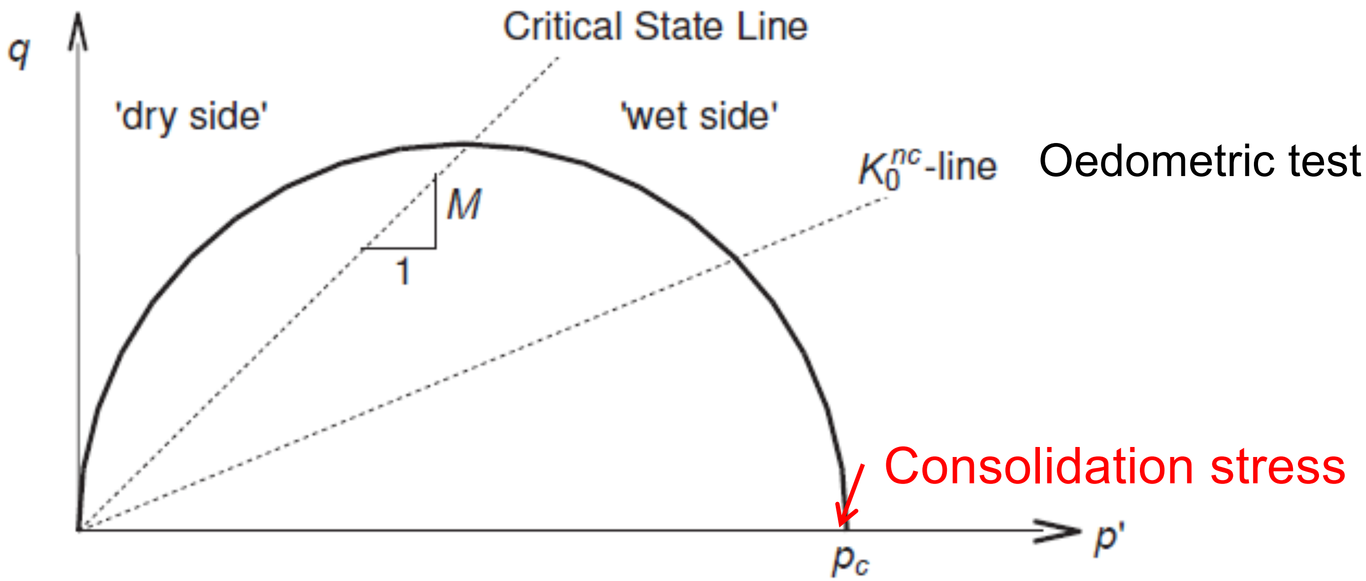


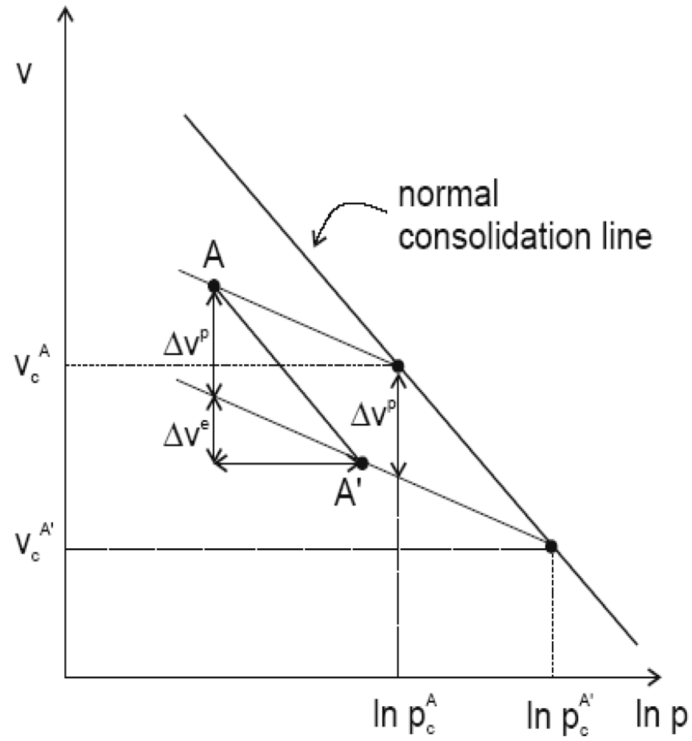
Figure 10.1 Yield surface of the Modified Cam-Clay model in p' - q-plane

Plastic potential function

Roscoe and Burland (1968) derived an associated plastic flow rule which describes the ratio between incremental plastic volumetric strain and incremental plastic shear strain. It is:

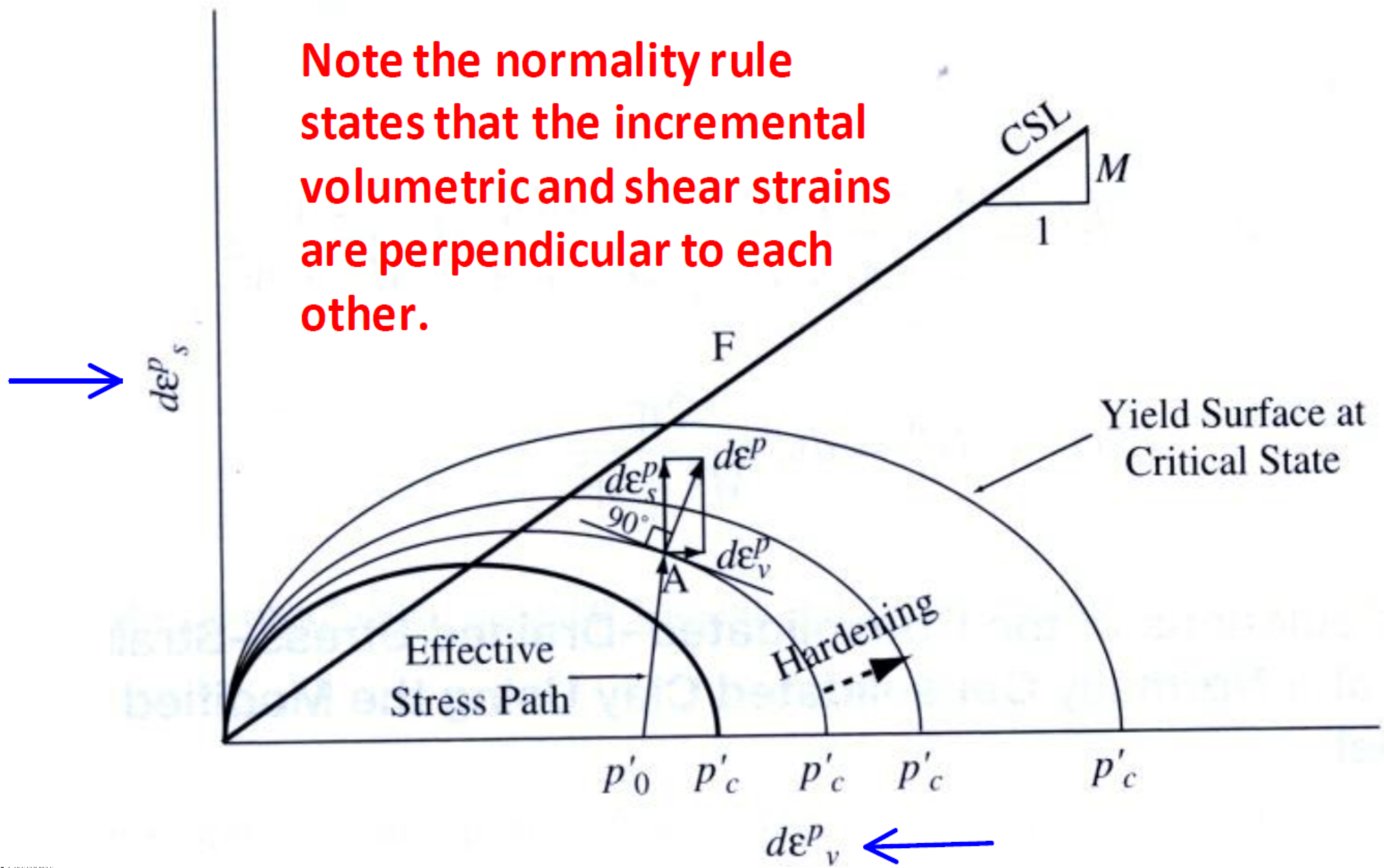
- $d\varepsilon_v^p / d\varepsilon_S^p = (M^2 - \eta^2) / 2\eta$

where $\eta = q/p'$ and at failure $\eta = M$

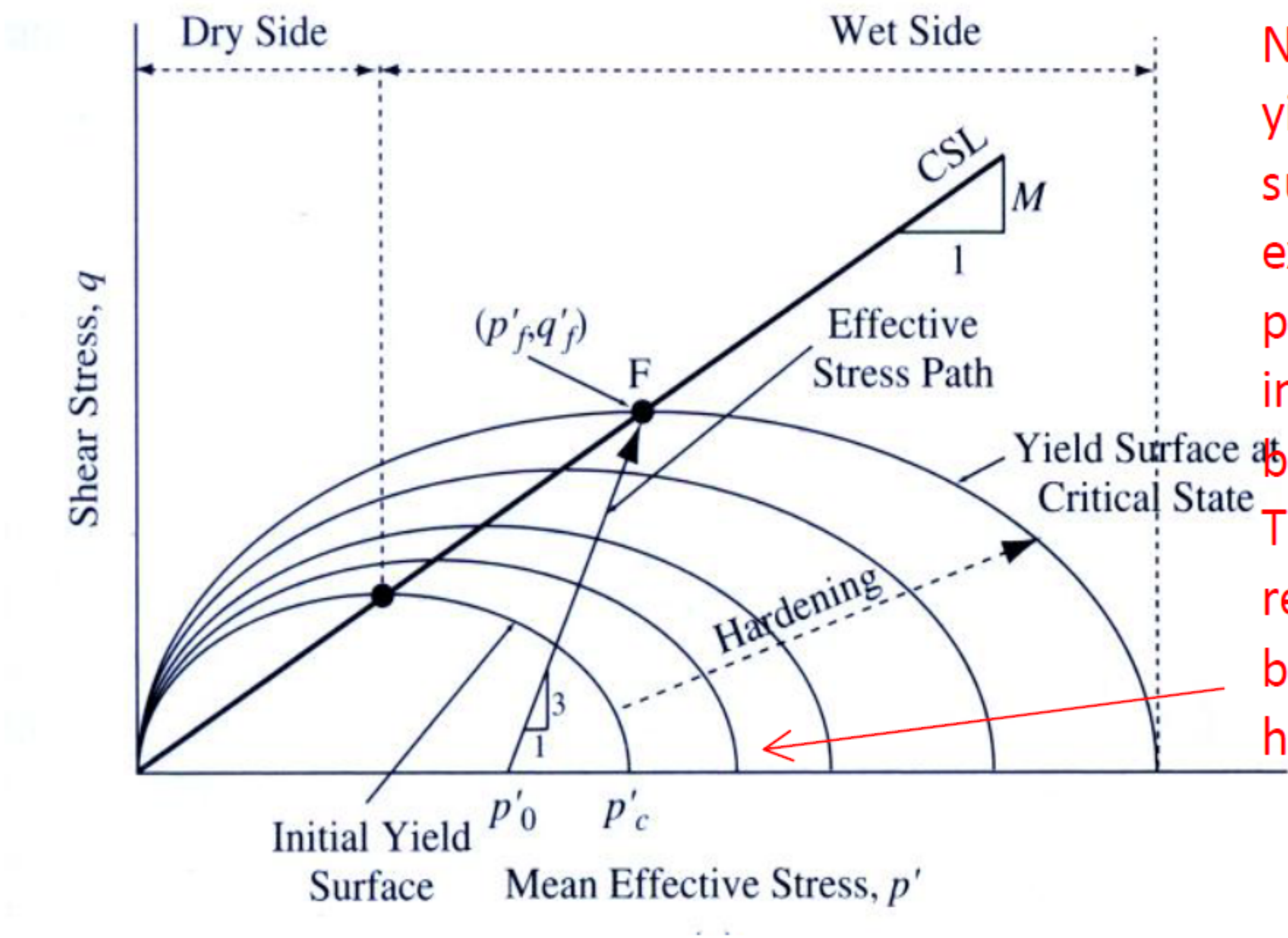


Hardening mechanism for normally or lightly overconsolidated clay

Note the normality rule states that the incremental volumetric and shear strains are perpendicular to each other.

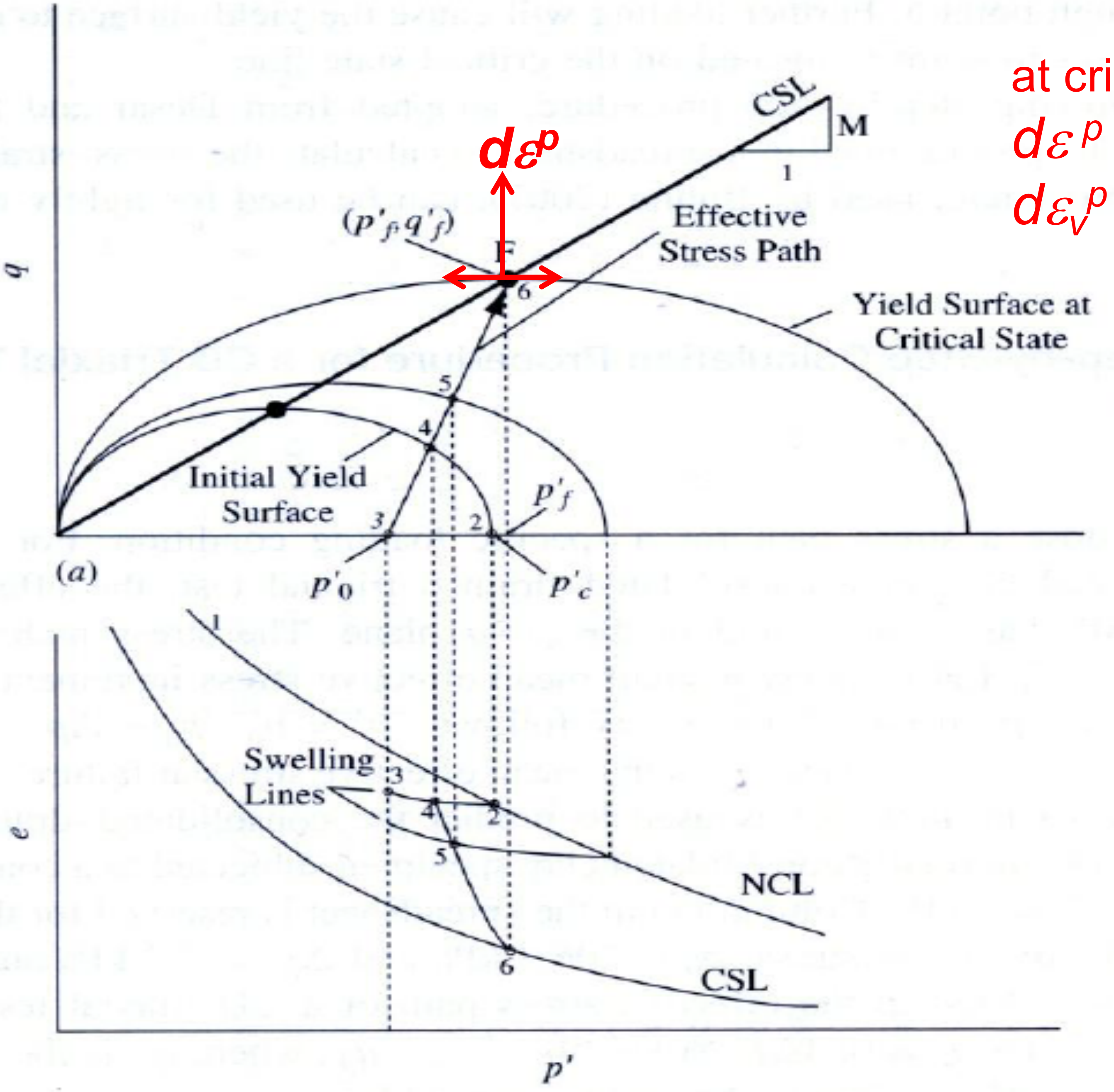


Strain hardening behavior for lightly overconsolidated clay



Note the yield surface expands as p' is increased beyond p'c. This is represented by strain hardening.

Consolidated Drained Test Behavior of Lightly Overconsolidated Clay

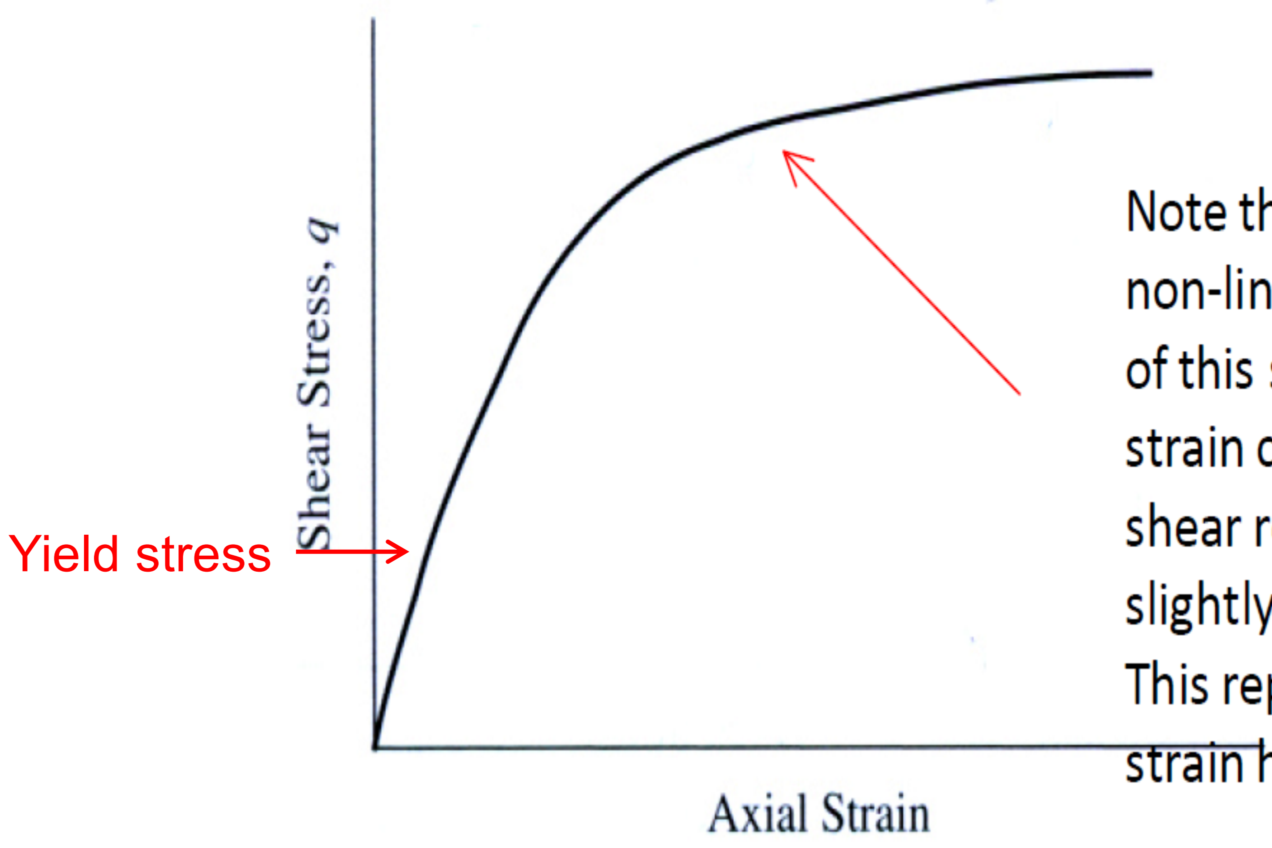


at critical state

$d\varepsilon^p = d\varepsilon_s^p$

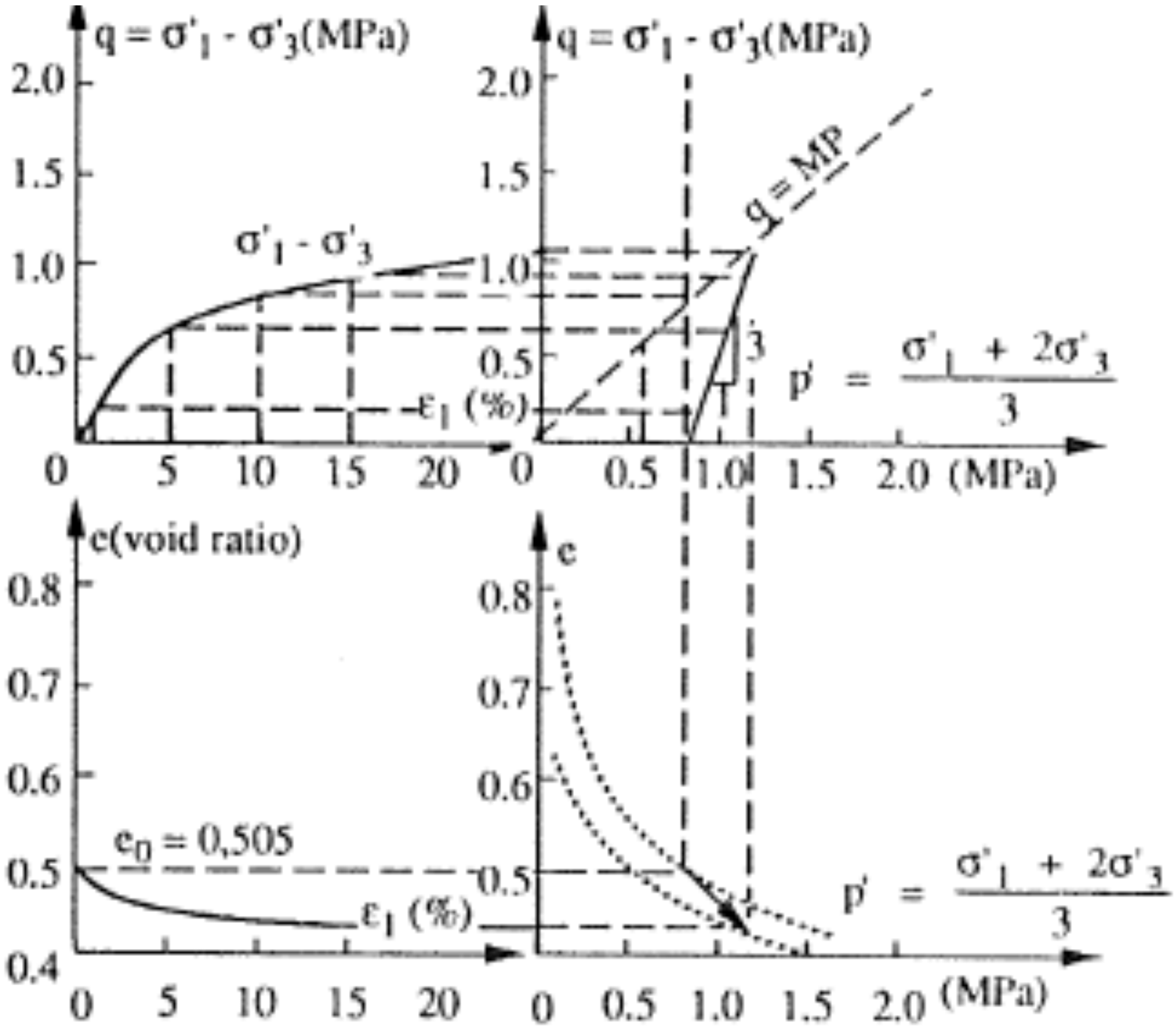
$d\varepsilon_v^p = 0$

Stress-Strain Curve showing strain hardening



Note that in the non-linear range of this stress-strain curve, the shear resistance is slightly increasing. This represents strain hardening.

Drained triaxial test on normally consolidated clay



In conclusion, the Modified Cam-Clay model is based on five parameters:

ν : Poisson's ratio

κ: Cam-Clay swelling index

λ: Cam-Clay compression index

M: Tangent of the critical state line

e_{init}: Initial void ratio

MMC hardening/softening law

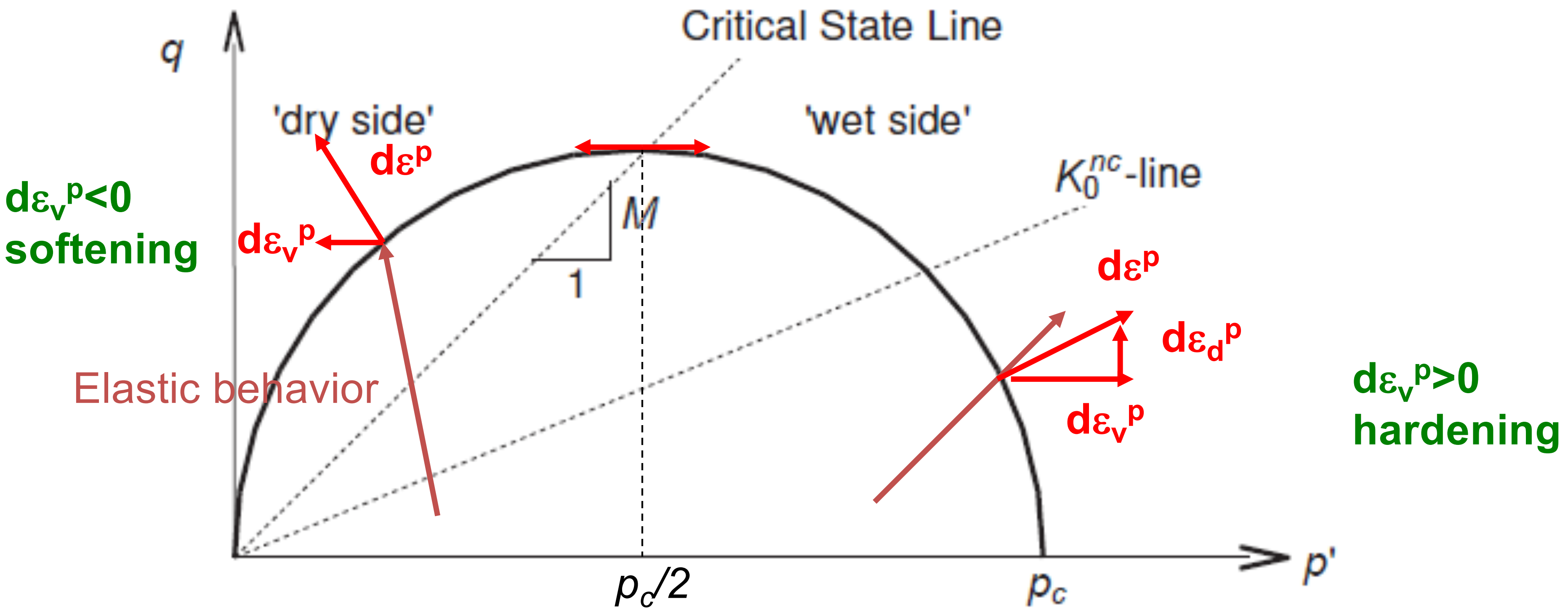
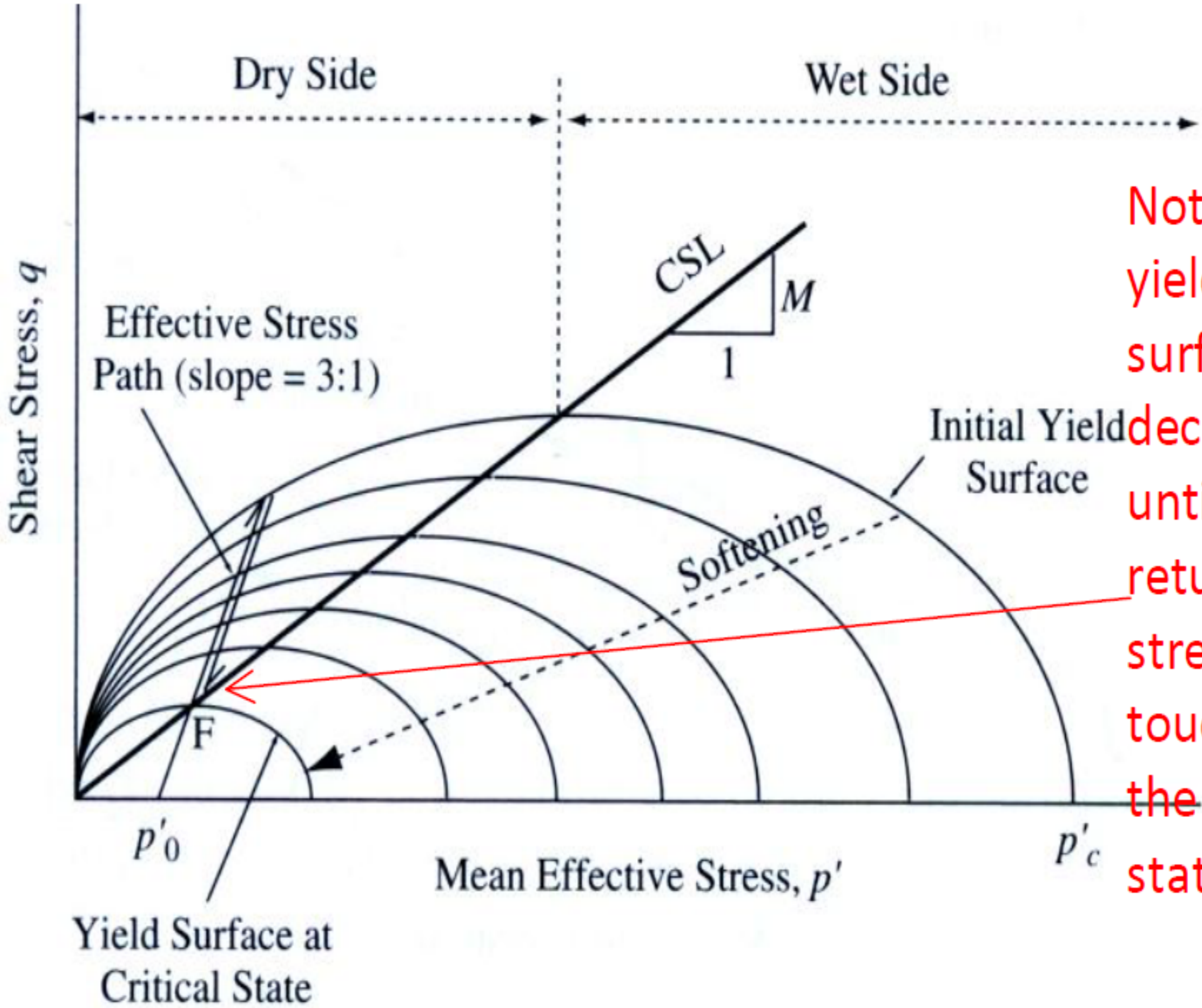


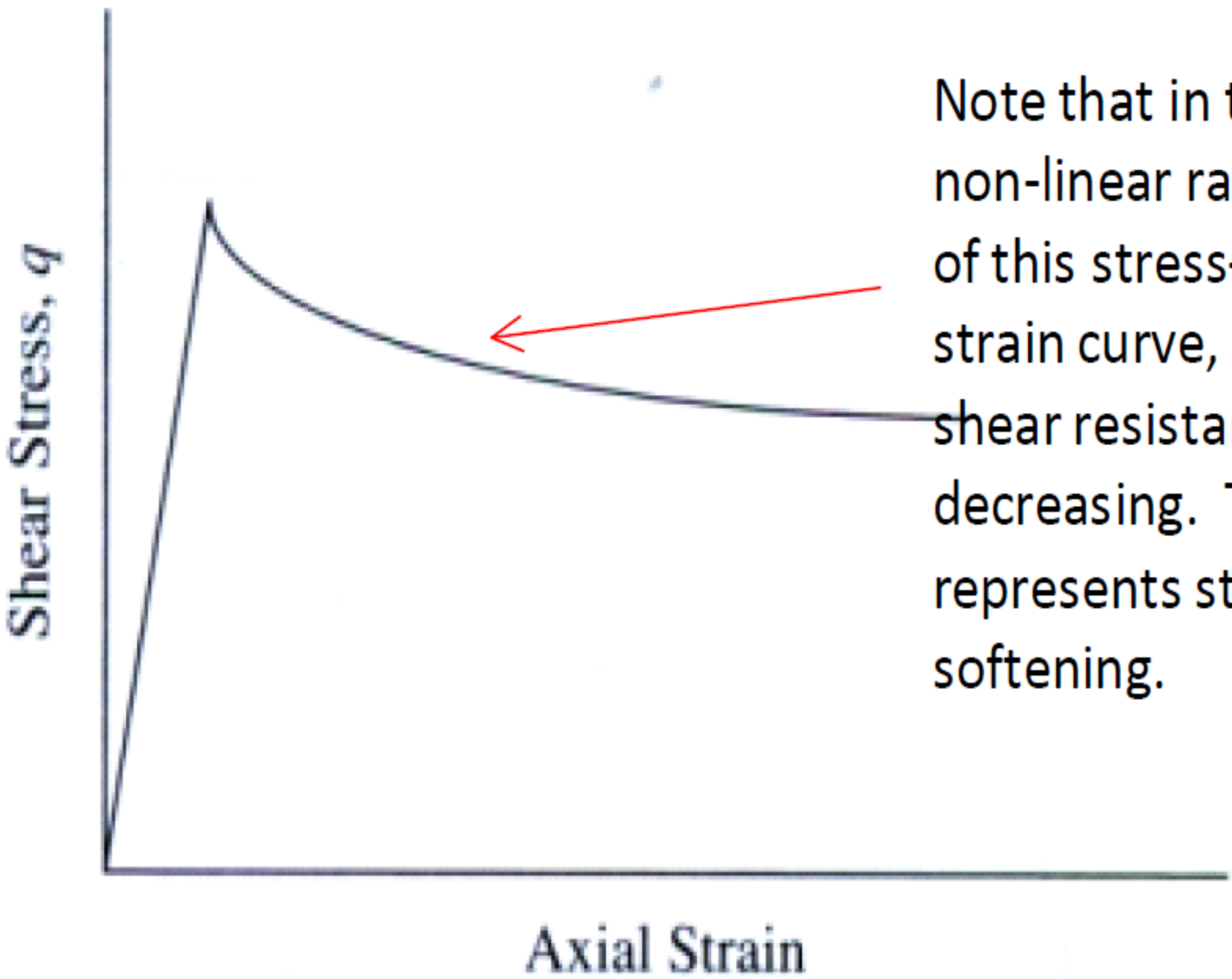
Figure 10.1 Yield surface of the Modified Cam-Clay model in p' - q-plane

Strain softening behavior for heavily overconsolidated clay



Note the yield surface decreases until the returning stress path touches the critical state line.

Stress-Strain Curve showing strain softening

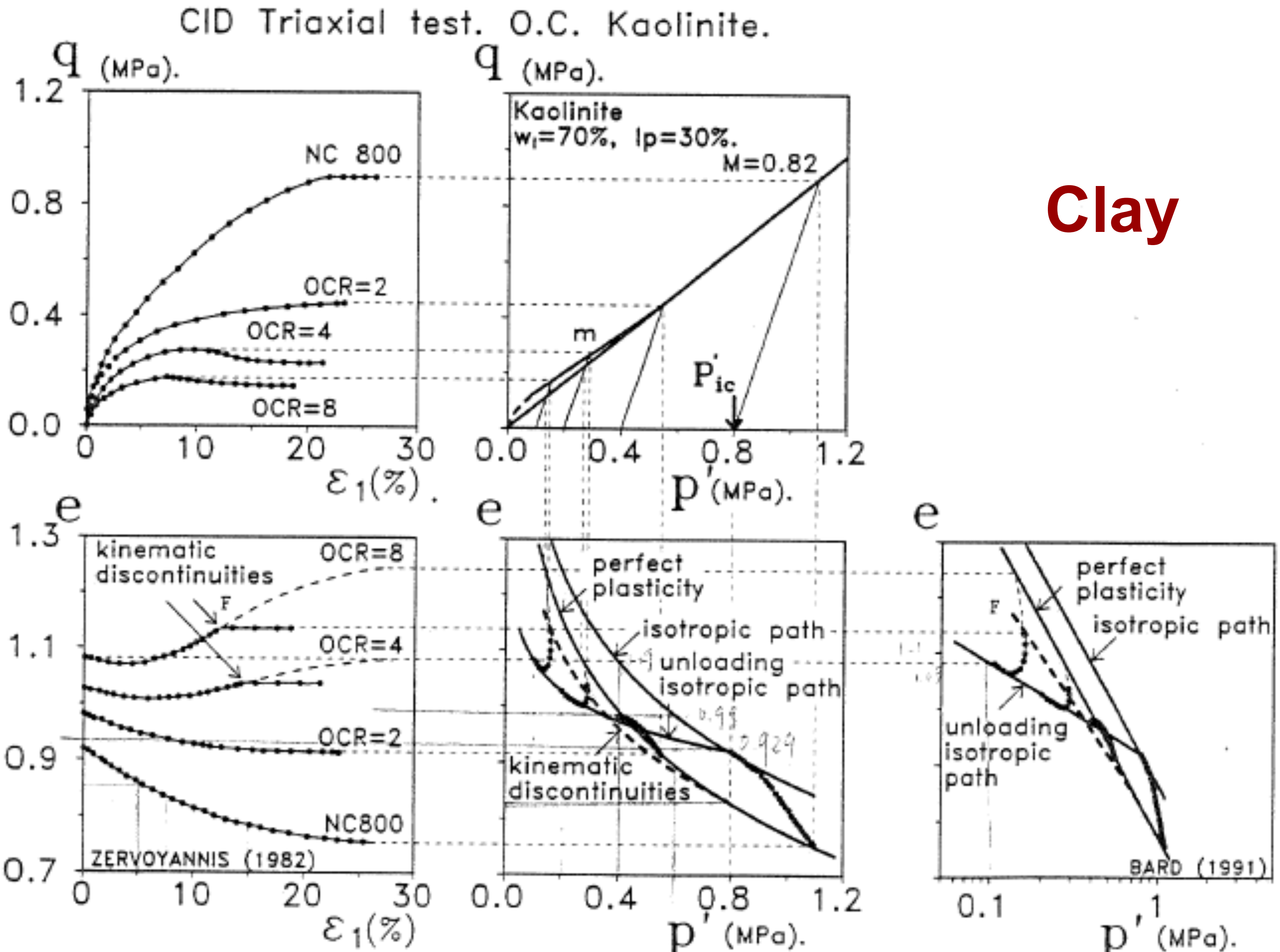


Note that in the non-linear range of this stress-strain curve, the shear resistance is decreasing. This represents strain softening.

Triaxial tests - Overconsolidated behavior

constant σ'_3

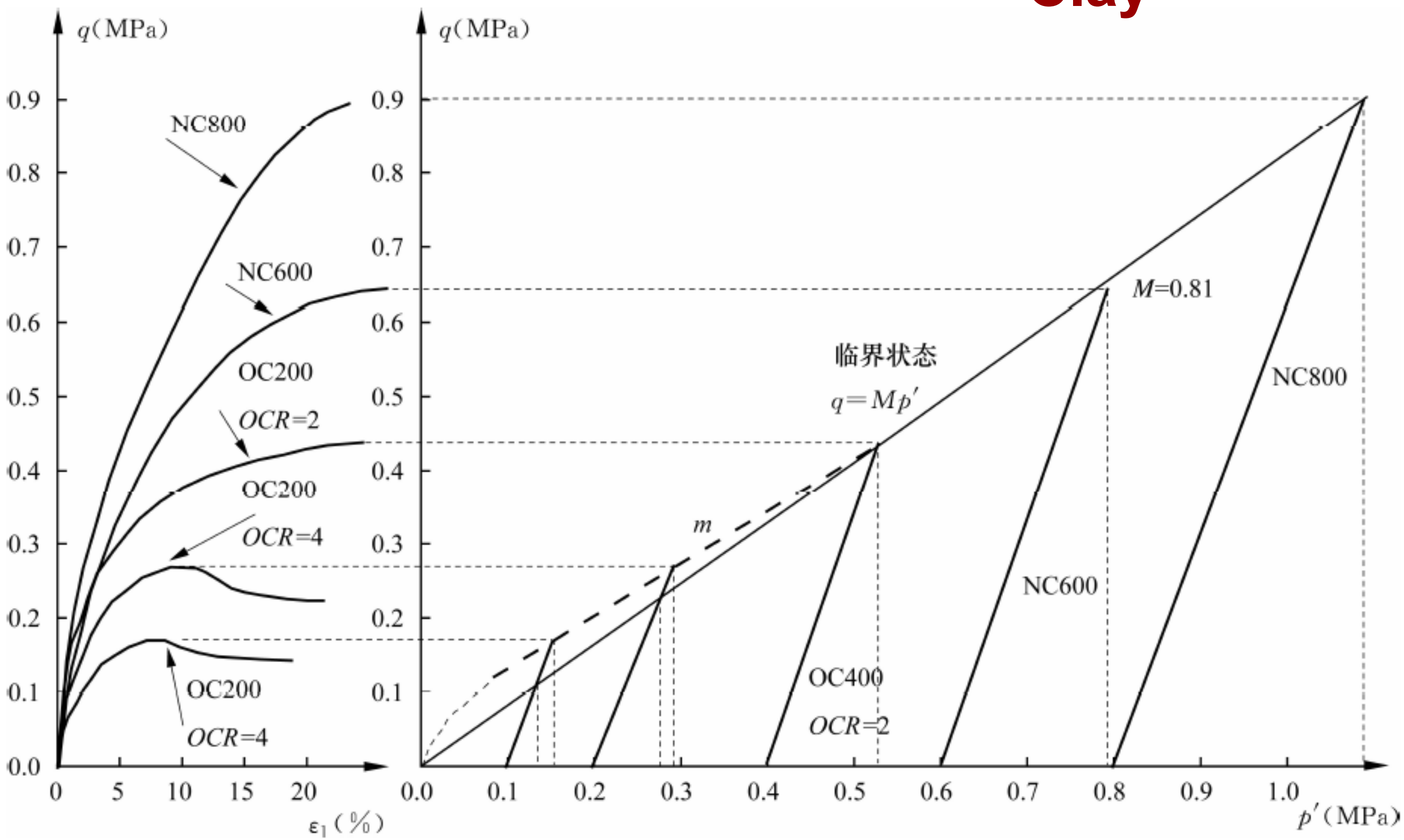
Clay



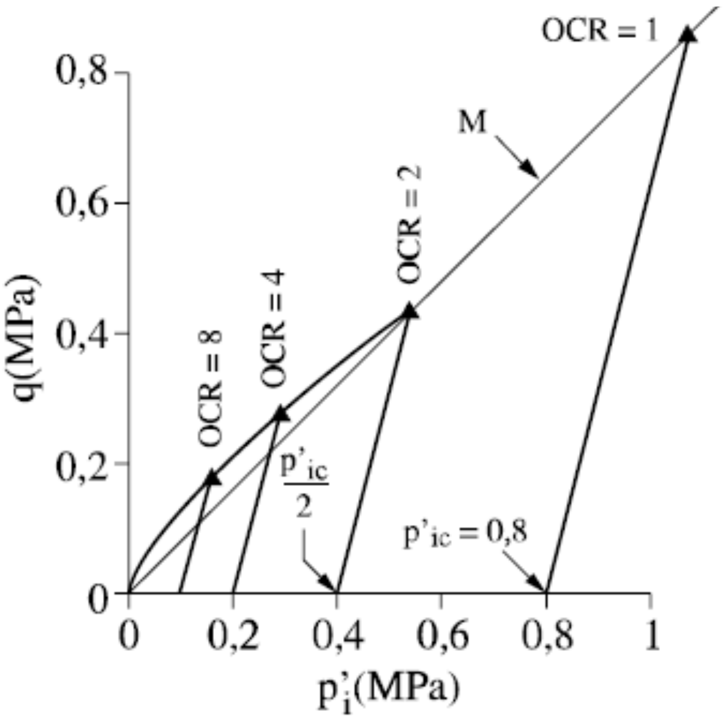
Triaxial tests - Overconsolidated behavior

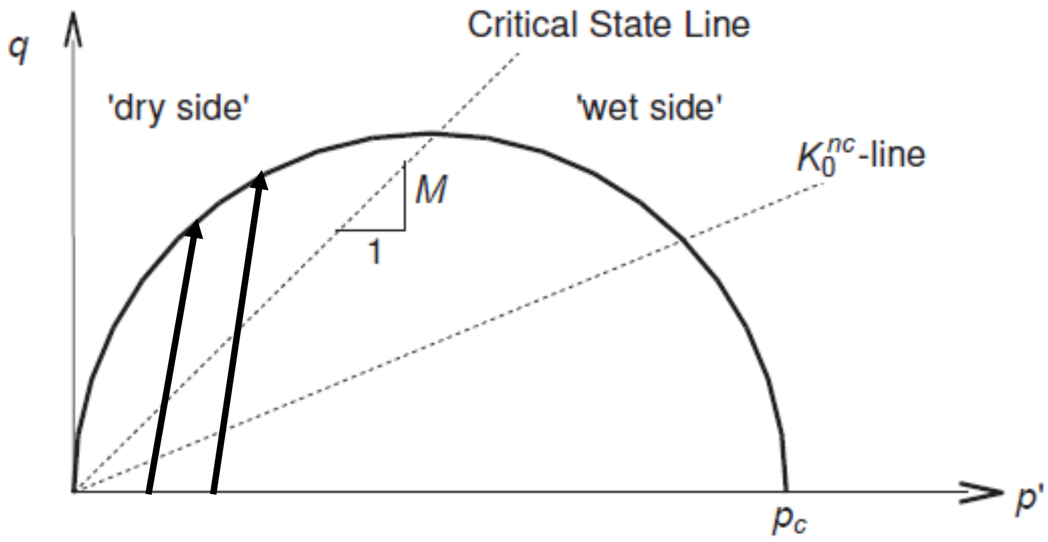
constant σ'_3

Clay



Maximum Strength Envelope: experiment/MCC Model





MCC not appropriate for overconsolidated clay
MCC not appropriate for sand

« What is the best model ? »

There is no direct answer to this question.

The universal model does not exist.

Each model has his advantages and its disadvantages.

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It is, therefore, necessary to be able to understand the **capabilities** as well as the **limitations** of the different models at our disposal in order to **select the appropriate one** according to the nature of the soil and the characteristics of the numerical simulations to be undertaken (foundations, tunnels, excavations,…).

This is not an easy task and there is no definite answer to this problem.

I will give you some elements of discussion and illustrations of the model performances for different cases

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Plaxis: Linear or Non-linear Elasticity

Stiffness moduli E_{50}^{ref}, E_{oed}^{ref} & E_{ur}^{ref} and power m

The advantage of the Hardening Soil model over the Mohr-Coulomb model is not only the use of a hyperbolic stress-strain curve instead of a bi-linear curve, but also the control of stress level dependency. When using the Mohr-Coulomb model, the user has to select a fixed value of Young's modulus whereas for real soils this stiffness depends on the stress level. It is therefore necessary to estimate the stress levels within the soil and use these to obtain suitable values of stiffness. With the Hardening Soil model, however, this cumbersome selection of input parameters is not required.

Instead, a stiffness modulus E_{50}^{ref} is defined for a reference minor principal effective stress of $-\sigma'_3 = p^{ref}$. As a default value, the program uses p^{ref} = 100 kN/m^2.

As some PLAXIS users are familiar with the input of shear moduli rather than the above stiffness moduli, shear moduli will now be discussed. Within Hooke's law of isotropic elasticity conversion between E and G goes by the equation $E = 2\,(1+\nu)\,G$. As E_{ur} is a

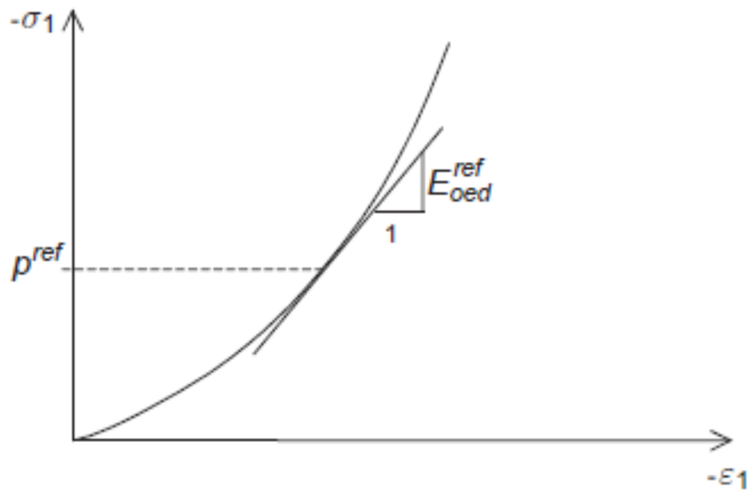

Figure 6.4 Definition of E_{oed}^{ref} in oedometer test results

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Plaxis : Loading or unloading

MC Model

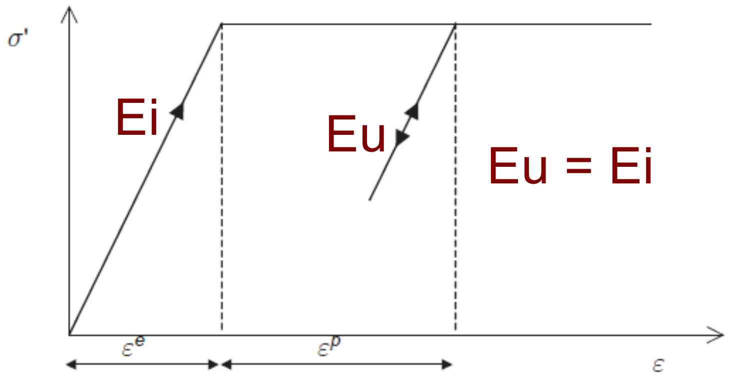
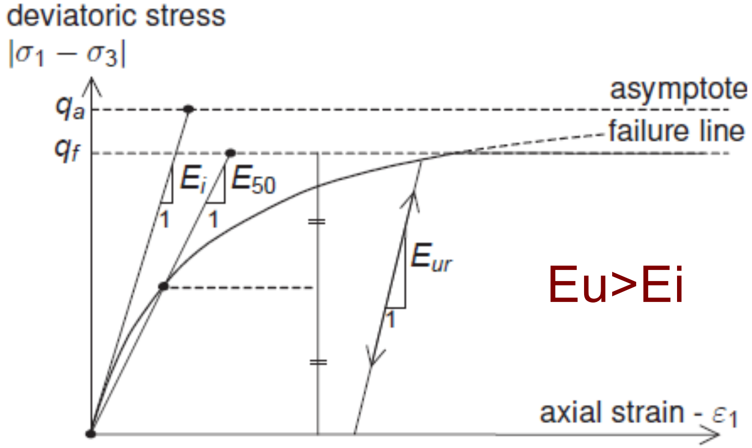
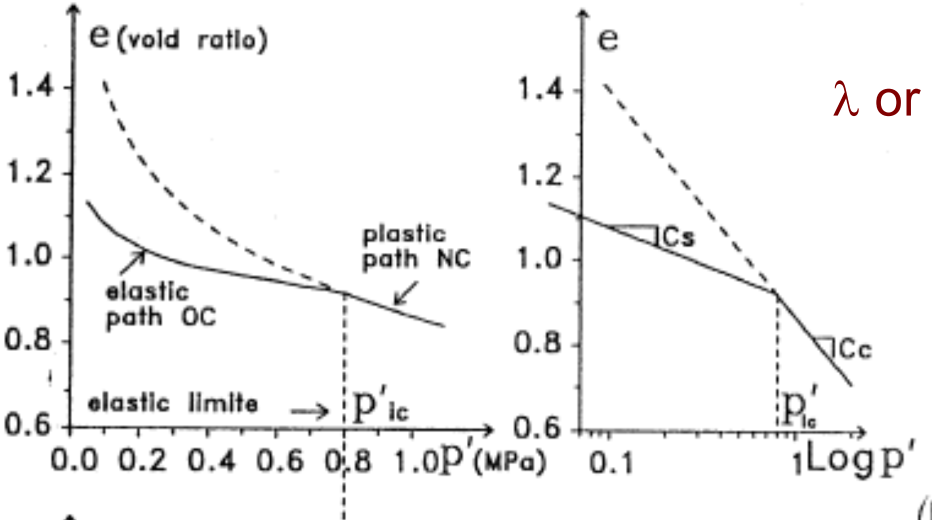


Figure 3.1 Basic idea of an elastic perfectly plastic model

HS Model



MCC Model



λ or c_c > κ or c_s

Example of an unloading case: soil excavation

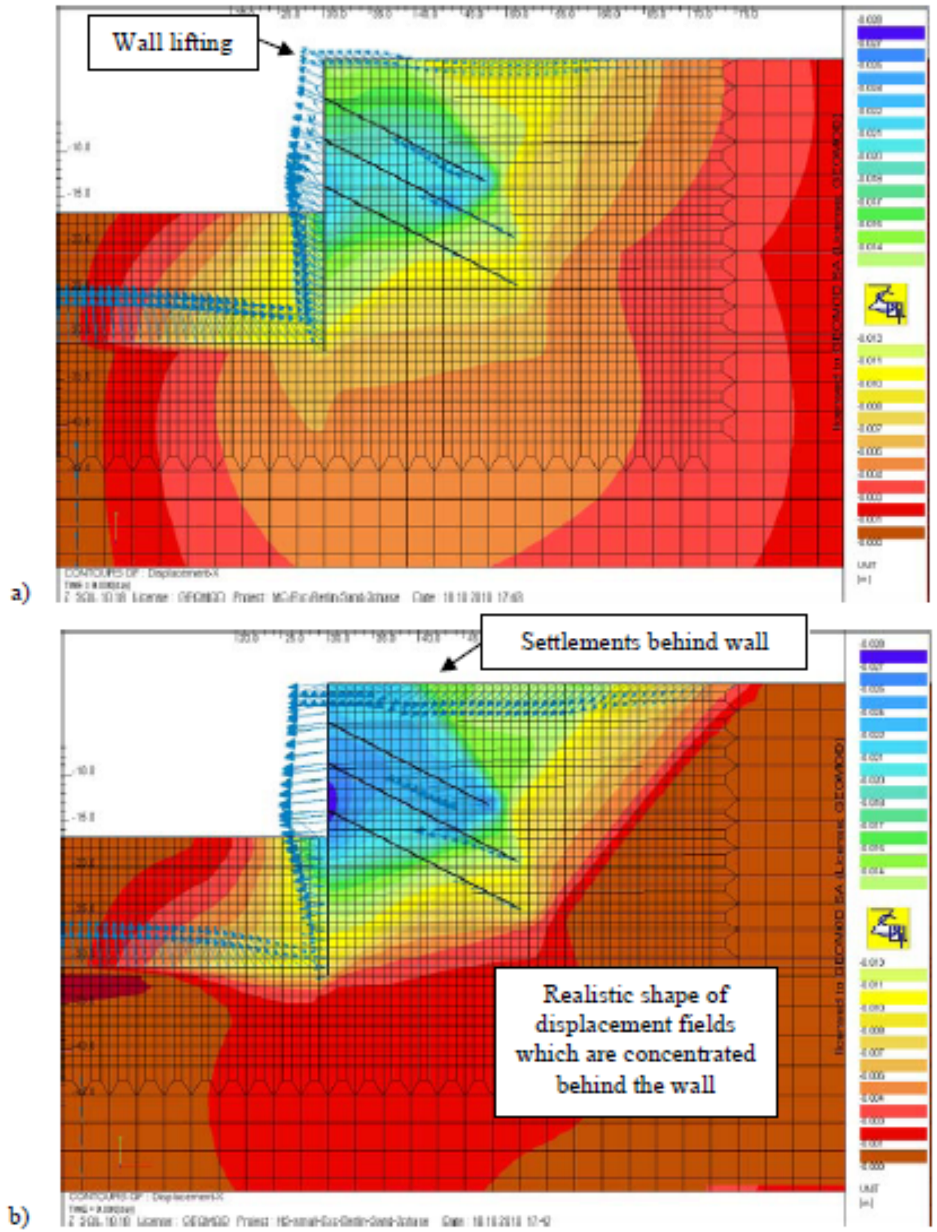


Figure 9 Comparison of numerical predictions of horizontal displacements for the excavation in Berlin Sand: (a) Mohr-Coulomb, (b) Hardening Soil SmallStrain.

Example of the influence of non-linear behaviour

Tunnel in London Clay

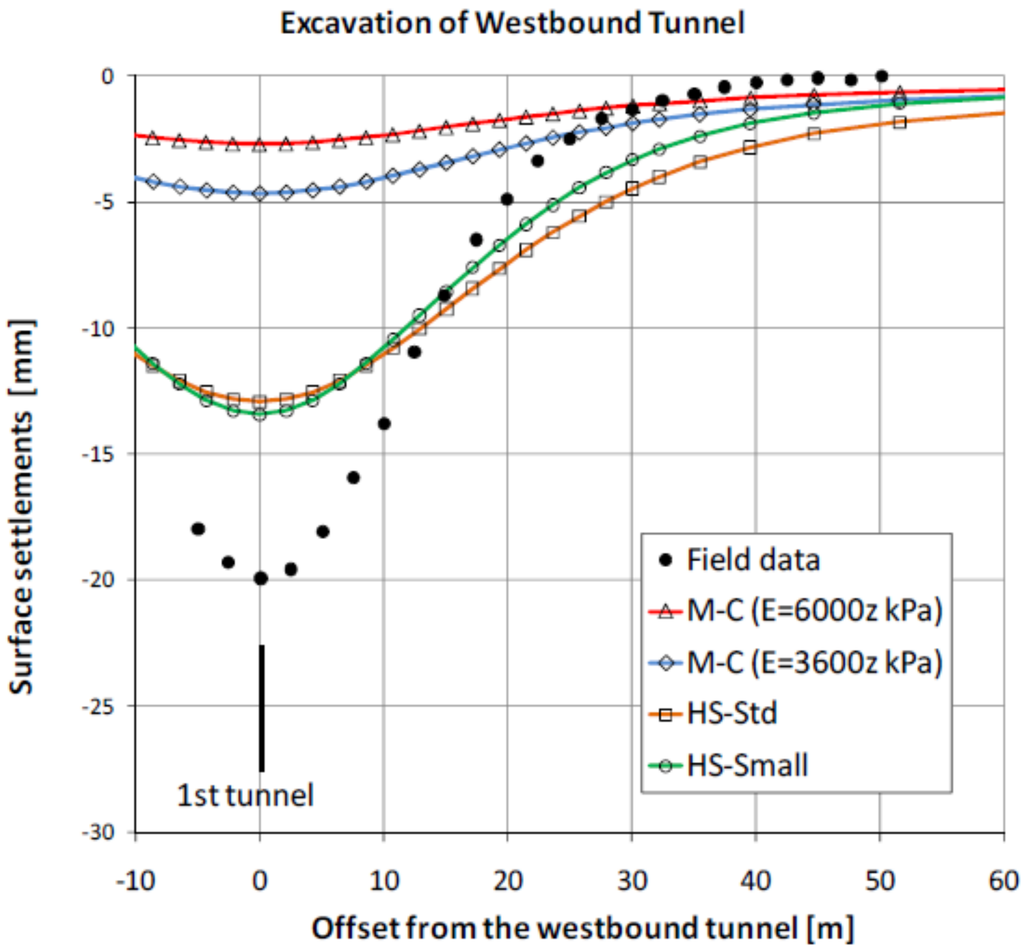


Figure 16 Surface settlement profiles after excavation of 1st tunnel: comparison for different models.

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Conclusion

The quality of the numerical simulations depends strongly not only on the choice of the constitutive model, but also on the parameter determination.

To improve this quality one should have

1. a thorough understanding of the capabilities and the limitations of a given model;
2. a well adapted procedure for the parameter identification

Ref. P.Y. Hicher & J.F. Shao (2008) “constitutive models for soils and rocks”, ed. ISTE-Wiley, 439 pages